Conformal structures with linear Fefferman-Graham equations

Paweł Nurowski

Centrum Fizyki Teoretycznej
Polska Akademia Nauk

Joint work with I. Anderson, Th. Leistner, A. Lischewski

Workshop Conformal Geometry and Spectral Theory
in honor of
ANDREAS JUHL

Berlin November 11-13, 2016

Plan

1. Ambient metrics and distributions
   - Fefferman-Graham construction
   - Ambient metrics for special conformal structures
2. Fefferman-Graham equations in terms of a perturbation h
   - Passing from g_ρ to $g + h_\rho$
3. Results
   - Theorems
   - New examples

Plan

1. Ambient metrics and distributions
   - Fefferman-Graham construction
   - Ambient metrics for special conformal structures
2. Fefferman-Graham equations in terms of a perturbation h
   - Passing from g_ρ to $g + h_\rho$
3. Results
   - Theorems
   - New examples

Ambient metric

- Let $(M^n, [g])$ be a conformal structure with metrics g of signature (n_+, n_-).
- An *ambient space* $\tilde{M}$ for $(M^n, [g])$ is locally a product

$$\tilde{M} =]0, +\infty[\times M^n \times]-\epsilon, \epsilon[, \qquad \epsilon > 0,$$

  with respective coordinates (t, x^i, ρ). Choose g from the conformal class of $[g]$. Then the *ambient metric* $\tilde{g}$ associated with (M^n, g) is an $(n_+ + 1, n_- + 1)$-signature metric on $\tilde{M}$ given by:

$$\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2 g(x^i, \rho)$$

  such that

$$g(x^i, \rho)_{|\rho=0} = g(x^i),$$

  and

$$Ric(\tilde{g}) = 0.$$

Ambient metric

- Let $(M^n, [g])$ be a conformal structure with metrics g of signature (n_+, n_-).
- An *ambient space* $\tilde{M}$ for $(M^n, [g])$ is locally a product

$$\tilde{M} =]0, +\infty[\times M^n \times]-\epsilon, \epsilon[, \qquad \epsilon > 0,$$

with respective coordinates (t, x^i, ρ). Choose g from the conformal class of $[g]$. Then the *ambient metric* $\tilde{g}$ associated with (M^n, g) is an $(n_+ + 1, n_- + 1)$-signature metric on $\tilde{M}$ given by:

$$\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2 g(x^i, \rho)$$

such that

$$g(x^i, \rho)_{|\rho=0} = g(x^i),$$

and

$$Ric(\tilde{g}) = 0.$$

Ambient metric

- Let $(M^n, [g])$ be a conformal structure with metrics g of signature (n_+, n_-).
- An *ambient space* $\tilde{M}$ for $(M^n, [g])$ is locally a product

$$\tilde{M} =]0, +\infty[\times M^n \times] - \epsilon, \epsilon[, \qquad \epsilon > 0,$$

with respective coordinates (t, x^i, ρ). Choose g from the conformal class of $[g]$. Then the *ambient metric* $\tilde{g}$ associated with (M^n, g) is an $(n_+ + 1, n_- + 1)$-signature metric on $\tilde{M}$ given by:

$$\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2 g(x^i, \rho)$$

such that

$$g(x^i, \rho)_{|\rho=0} = g(x^i),$$

and

$$Ric(\tilde{g}) = 0.$$

Ambient metric

- Let $(M^n, [g])$ be a conformal structure with metrics g of signature (n_+, n_-).
- An *ambient space* $\tilde{M}$ for $(M^n, [g])$ is locally a product

$$\tilde{M} =]0, +\infty[\times M^n \times]-\epsilon, \epsilon[, \qquad \epsilon > 0,$$

with respective coordinates (t, x^i, ρ). Choose g from the conformal class of $[g]$. Then the *ambient metric* $\tilde{g}$ associated with (M^n, g) is an $(n_+ + 1, n_- + 1)$-signature metric on $\tilde{M}$ given by:

$$\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2 g(x^i, \rho)$$

such that

$$g(x^i, \rho)_{|\rho=0} = g(x^i),$$

and

$$Ric(\tilde{g}) = 0.$$

Ambient metric

- Let $(M^n, [g])$ be a conformal structure with metrics g of signature (n_+, n_-).
- An *ambient space* $\tilde{M}$ for $(M^n, [g])$ is locally a product

$$\tilde{M} =]0, +\infty[\times M^n \times] - \epsilon, \epsilon[, \qquad \epsilon > 0,$$

with respective coordinates (t, x^i, ρ). Choose g from the conformal class of $[g]$. Then the *ambient metric* $\tilde{g}$ associated with (M^n, g) is an $(n_+ + 1, n_- + 1)$-signature metric on $\tilde{M}$ given by:

$$\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2 g(x^i, \rho)$$

such that

$$g(x^i, \rho)_{|\rho=0} = g(x^i),$$

and

$$Ric(\tilde{g}) = 0.$$

Ambient metric

- Let $(M^n, [g])$ be a conformal structure with metrics g of signature (n_+, n_-).
- An *ambient space* $\tilde{M}$ for $(M^n, [g])$ is locally a product

$$\tilde{M} =]0, +\infty[\times M^n \times] - \epsilon, \epsilon[, \qquad \epsilon > 0,$$

with respective coordinates (t, x^i, ρ). Choose g from the conformal class of $[g]$. Then the *ambient metric* $\tilde{g}$ associated with (M^n, g) is an $(n_+ + 1, n_- + 1)$-signature metric on $\tilde{M}$ given by:

$$\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2 g(x^i, \rho)$$

such that

$$g(x^i, \rho)_{|\rho=0} = g(x^i),$$

and

$$Ric(\tilde{g}) = 0.$$

Explicit ambient metrics?

- If $[g]$ contains an *Einstein* metric g_0, $Ric(g_0) = \Lambda g_0$, then

$$\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2(1 + \frac{\Lambda\rho}{2(n-1)})^2 g_0.$$

- But otherwise finding explicit examples of ambient metrics is difficult.

Explicit ambient metrics?

- If $[g]$ contains an *Einstein* metric g_0, $Ric(g_0) = \Lambda g_0$, then

$$\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2 \left(1 + \frac{\Lambda \rho}{2(n-1)}\right)^2 g_0.$$

- But otherwise finding explicit examples of ambient metrics is difficult.

Explicit ambient metrics?

- If $[g]$ contains an *Einstein* metric g_0, $Ric(g_0) = \Lambda g_0$, then

$$\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2(1 + \frac{\Lambda\rho}{2(n-1)})^2 g_0.$$

- But otherwise finding explicit examples of ambient metrics is difficult.

Explicit ambient metrics?

- If $[g]$ contains an *Einstein* metric g_0, $Ric(g_0) = \Lambda g_0$, then

$$\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2\left(1 + \frac{\Lambda\rho}{2(n-1)}\right)^2 g_0.$$

- But otherwise finding explicit examples of ambient metrics is difficult.

Explicit ambient metrics?

- Write $g(x^k,\rho)=g_{ij}\mathrm{d}x^i\mathrm{d}x^j$ then $Ric(\tilde{g})=0$ for $\tilde{g}=2\mathrm{d}t\mathrm{d}(\rho t)+t^2g(x^k,\rho)$ is equivalent to the following system of PDE's:

$$\rho\ddot{g}_{ij}-(\frac{n}{2}-1)\dot{g}_{ij}-\rho g^{kl}\dot{g}_{ik}\dot{g}_{jl}+\tfrac{1}{2}\rho g^{kl}\dot{g}_{kl}\dot{g}_{ij}-\tfrac{1}{2}g^{kl}\dot{g}_{kl}g_{ij}+R_{ij}=0,$$
$$g^{kl}\left(\nabla_k\dot{g}_{il}-\nabla_i\dot{g}_{kl}\right)=0,$$
$$g^{kl}\ddot{g}_{kl}+\tfrac{1}{2}g^{kl}g^{pq}\dot{g}_{pk}\dot{g}_{ql}=0.$$

- Here 'dot' denotes differentiation w. r. t. ρ, ∇ is Levi-Civita connection for $g_\rho=g(x^k,\rho)$, and R_{ij} is the Ricci tensor for g_ρ.

Explicit ambient metrics?

- Write $g(x^k,\rho) = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$ then $Ric(\tilde g) = 0$ for $\tilde g = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2 g(x^k,\rho)$ is equivalent to the following system of PDE's:

$$\rho\ddot g_{ij} - (\frac{n}{2} - 1)\dot g_{ij} - \rho g^{kl}\dot g_{ik}\dot g_{jl} + \tfrac{1}{2}\rho g^{kl}\dot g_{kl}\dot g_{ij} - \tfrac{1}{2}g^{kl}\dot g_{kl}g_{ij} + R_{ij} = 0,$$
$$g^{kl}\left(\nabla_k\dot g_{il} - \nabla_i\dot g_{kl}\right) = 0,$$
$$g^{kl}\ddot g_{kl} + \tfrac{1}{2}g^{kl}g^{pq}\dot g_{pk}\dot g_{ql} = 0.$$

- Here 'dot' denotes differentiation w. r. t. ρ, ∇ is Levi-Civita connection for $g_\rho = g(x^k,\rho)$, and R_{ij} is the Ricci tensor for g_ρ.

Explicit ambient metrics?

- Write $g(x^k,\rho) = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$ then $Ric(\tilde{g}) = 0$ for $\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2 g(x^k,\rho)$ is equivalent to the following system of PDE's:

$$\rho\ddot{g}_{ij} - (\frac{n}{2} - 1)\dot{g}_{ij} - \rho g^{kl}\dot{g}_{ik}\dot{g}_{jl} + \tfrac{1}{2}\rho g^{kl}\dot{g}_{kl}\dot{g}_{ij} - \tfrac{1}{2}g^{kl}\dot{g}_{kl}g_{ij} + R_{ij} = 0,$$
$$g^{kl}(\nabla_k\dot{g}_{il} - \nabla_i\dot{g}_{kl}) = 0,$$
$$g^{kl}\ddot{g}_{kl} + \tfrac{1}{2}g^{kl}g^{pq}\dot{g}_{pk}\dot{g}_{ql} = 0.$$

- Here 'dot' denotes differentiation w. r. t. ρ, ∇ is Levi-Civita connection for $g_\rho = g(x^k,\rho)$, and R_{ij} is the Ricci tensor for g_ρ.

Explicit ambient metrics?

- Write $g(x^k,\rho) = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$ then $Ric(\tilde{g}) = 0$ for $\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2 g(x^k,\rho)$ is equivalent to the following system of PDE's:

$$\rho\ddot{g}_{ij} - (\frac{n}{2} - 1)\dot{g}_{ij} - \rho g^{kl}\dot{g}_{ik}\dot{g}_{jl} + \tfrac{1}{2}\rho g^{kl}\dot{g}_{kl}\dot{g}_{ij} - \tfrac{1}{2}g^{kl}\dot{g}_{kl}g_{ij} + R_{ij} = 0,$$
$$g^{kl}\left(\nabla_k\dot{g}_{il} - \nabla_i\dot{g}_{kl}\right) = 0,$$
$$g^{kl}\ddot{g}_{kl} + \tfrac{1}{2}g^{kl}g^{pq}\dot{g}_{pk}\dot{g}_{ql} = 0.$$

- Here 'dot' denotes differentiation w. r. t. ρ, ∇ is Levi-Civita connection for $g_\rho = g(x^k,\rho)$, and R_{ij} is the Ricci tensor for g_ρ.

Plan

1. Ambient metrics and distributions
   - Fefferman-Graham construction
   - Ambient metrics for special conformal structures
2. Fefferman-Graham equations in terms of a perturbation h
   - Passing from g_ρ to $g + h_\rho$
3. Results
   - Theorems
   - New examples

Ambient metrics for my favorite conformal structure

- A distribution $\mathcal{D}$ on a 5-manifold M^5 is called $(2, 3, 5)$ if
  - $\mathcal{D}$ has rank 2,
  - $\mathcal{D} + [\mathcal{D}, \mathcal{D}]$ has rank 3, and
  - $\mathcal{D} + [\mathcal{D}, \mathcal{D}] + [\mathcal{D}, [\mathcal{D}, \mathcal{D}]]$ has rank 5.
- Every $(2, 3, 5)$ distribution $\mathcal{D}$ canonically defines a $(3, 2)$ signature conformal structure $[g_{\mathcal{D}}]$ on M^5, which encodes the geometry of the distribution.

Ambient metrics for my favorite conformal structure

- A distribution $\mathcal{D}$ on a 5-manifold M^5 is called $(2, 3, 5)$ if
  - $\mathcal{D}$ has rank 2,
  - $\mathcal{D} + [\mathcal{D}, \mathcal{D}]$ has rank 3, and
  - $\mathcal{D} + [\mathcal{D}, \mathcal{D}] + [\mathcal{D}, [\mathcal{D}, \mathcal{D}]]$ has rank 5.
- Every $(2, 3, 5)$ distribution $\mathcal{D}$ canonically defines a $(3, 2)$ signature conformal structure $[g_{\mathcal{D}}]$ on M^5, which encodes the geometry of the distribution.

Ambient metrics for my favorite conformal structure

- A distribution $\mathcal{D}$ on a 5-manifold M^5 is called $(2, 3, 5)$ if
  - $\mathcal{D}$ has rank 2,
  - $\mathcal{D} + [\mathcal{D}, \mathcal{D}]$ has rank 3, and
  - $\mathcal{D} + [\mathcal{D}, \mathcal{D}] + [\mathcal{D}, [\mathcal{D}, \mathcal{D}]]$ has rank 5.
- Every $(2, 3, 5)$ distribution $\mathcal{D}$ canonically defines a $(3, 2)$ signature conformal structure $[g_{\mathcal{D}}]$ on M^5, which encodes the geometry of the distribution.

Ambient metrics for my favorite conformal structure

- A distribution $\mathcal{D}$ on a 5-manifold M^5 is called $(2, 3, 5)$ if
  - $\mathcal{D}$ has rank 2,
  - $\mathcal{D} + [\mathcal{D}, \mathcal{D}]$ has rank 3, and
  - $\mathcal{D} + [\mathcal{D}, \mathcal{D}] + [\mathcal{D}, [\mathcal{D}, \mathcal{D}]]$ has rank 5.
- Every $(2, 3, 5)$ distribution $\mathcal{D}$ canonically defines a $(3, 2)$ signature conformal structure $[g_{\mathcal{D}}]$ on M^5, which encodes the geometry of the distribution.

Ambient metrics for my favorite conformal structure

- A distribution $\mathcal{D}$ on a 5-manifold M^5 is called $(2, 3, 5)$ if
  - $\mathcal{D}$ has rank 2,
  - $\mathcal{D} + [\mathcal{D}, \mathcal{D}]$ has rank 3, and
  - $\mathcal{D} + [\mathcal{D}, \mathcal{D}] + [\mathcal{D}, [\mathcal{D}, \mathcal{D}]]$ has rank 5.
- Every $(2, 3, 5)$ distribution $\mathcal{D}$ canonically defines a $(3, 2)$ signature conformal structure $[g_{\mathcal{D}}]$ on M^5, which encodes the geometry of the distribution.

Ambient metrics for my favorite conformal structure

- A distribution $\mathcal{D}$ on a 5-manifold M^5 is called $(2, 3, 5)$ if
  - $\mathcal{D}$ has rank 2,
  - $\mathcal{D} + [\mathcal{D}, \mathcal{D}]$ has rank 3, and
  - $\mathcal{D} + [\mathcal{D}, \mathcal{D}] + [\mathcal{D}, [\mathcal{D}, \mathcal{D}]]$ has rank 5.
- Every $(2, 3, 5)$ distribution $\mathcal{D}$ canonically defines a $(3, 2)$ signature conformal structure $[g_{\mathcal{D}}]$ on M^5, which encodes the geometry of the distribution.

Ambient metrics for my favorite conformal structure

- For example a distribution

$$\mathcal{D} = (\partial_q, \partial_x + p\partial_y + q\partial_p + F\partial_z)$$

is $(2,3,5)$ iff the function $F = F(x,y,p,q,z)$ satisfies $F_{qq} \neq 0$.

- Taking $F = q^2 + f(x,p) + bz$ with $b = \mathrm{const}$, the conformal class $[g_\mathcal{D}]$ may be represented by a metric $g_\mathcal{D}$ in a relatively simple form:

$$g_{\mathcal{D}} = 8\Big(\mathrm{d}p - q\mathrm{d}x\Big)^2 - 6\Big(\mathrm{d}z - 2q\mathrm{d}p + (q^2 - f - bz)\mathrm{d}x\Big)\mathrm{d}x -$$

$$2\Big(\mathrm{d}y - p\mathrm{d}x\Big)\Big(6\mathrm{d}q - 2b\mathrm{d}p - (\tfrac{2}{5}b^2 + \tfrac{9}{10}f_{pp})(\mathrm{d}y - p\mathrm{d}x) - (4bq + 3f_p)\mathrm{d}x\Big).$$

Ambient metrics for my favorite conformal structure

- For example a distribution

$$\mathcal{D} = (\partial_q, \partial_x + p\partial_y + q\partial_p + F\partial_z)$$

is $(2,3,5)$ iff the function $F = F(x,y,p,q,z)$ satisfies $F_{qq} \neq 0$.

- Taking $F = q^2 + f(x,p) + bz$ with $b = \mathrm{const}$, the conformal class $[g_{\mathcal{D}}]$ may be represented by a metric $g_{\mathcal{D}}$ in a relatively simple form:

$$g_{\mathcal{D}} = 8\Big(\mathrm{d}p - q\mathrm{d}x\Big)^2 - 6\Big(\mathrm{d}z - 2q\mathrm{d}p + (q^2 - f - bz)\mathrm{d}x\Big)\mathrm{d}x -$$
$$2\Big(\mathrm{d}y - p\mathrm{d}x\Big)\Big(6\mathrm{d}q - 2b\mathrm{d}p - (\tfrac{2}{5}b^2 + \tfrac{9}{10}f_{pp})(\mathrm{d}y - p\mathrm{d}x) - (4bq + 3f_p)\mathrm{d}x\Big).$$

Ambient metrics for my favorite conformal structure

- For example a distribution

$$\mathcal{D} = (\partial_q, \partial_x + p\partial_y + q\partial_p + F\partial_z)$$

  is $(2,3,5)$ iff the function $F = F(x,y,p,q,z)$ satisfies $F_{qq} \neq 0$.

- Taking $F = q^2 + f(x,p) + bz$ with $b = \mathrm{const}$, the conformal class $[g_{\mathcal{D}}]$ may be represented by a metric $g_{\mathcal{D}}$ in a relatively simple form:

$$g_{\mathcal{D}} = 8\Big(\mathrm{d}p - q\mathrm{d}x\Big)^2 - 6\Big(\mathrm{d}z - 2q\mathrm{d}p + (q^2 - f - bz)\mathrm{d}x\Big)\mathrm{d}x -$$
$$2\Big(\mathrm{d}y - p\mathrm{d}x\Big)\Big(6\mathrm{d}q - 2b\mathrm{d}p - (\tfrac{2}{5}b^2 + \tfrac{9}{10}f_{pp})(\mathrm{d}y - p\mathrm{d}x) - (4bq + 3f_p)\mathrm{d}x\Big).$$

Ambient metrics for my favorite conformal structure

- For example a distribution

$$\mathcal{D} = (\partial_q, \partial_x + p\partial_y + q\partial_p + F\partial_z)$$

is $(2,3,5)$ iff the function $F = F(x,y,p,q,z)$ satisfies $F_{qq} \neq 0$.

- Taking $F = q^2 + f(x,p) + bz$ with $b = \mathrm{const}$, the conformal class $[g_{\mathcal{D}}]$ may be represented by a metric $g_{\mathcal{D}}$ in a relatively simple form:

$$g_{\mathcal{D}} = 8\Big(\mathrm{d}p - q\mathrm{d}x\Big)^2 - 6\Big(\mathrm{d}z - 2q\mathrm{d}p + (q^2 - f - bz)\mathrm{d}x\Big)\mathrm{d}x - 2\Big(\mathrm{d}y - p\mathrm{d}x\Big)\Big(6\mathrm{d}q - 2b\mathrm{d}p - (\tfrac{2}{5}b^2 + \tfrac{9}{10}f_{pp})(\mathrm{d}y - p\mathrm{d}x) - (4bq + 3f_p)\mathrm{d}x\Big).$$

Ambient metrics for my favorite conformal structure

- For example a distribution

$$\mathcal{D} = (\partial_q, \partial_x + p\partial_y + q\partial_p + F\partial_z)$$

 is $(2,3,5)$ iff the function $F = F(x,y,p,q,z)$ satisfies $F_{qq} \neq 0$.

- Taking $F = q^2 + f(x,p) + bz$ with $b = \mathrm{const}$, the conformal class $[g_{\mathcal{D}}]$ may be represented by a metric $g_{\mathcal{D}}$ in a relatively simple form:

$$g_{\mathcal{D}}{=}8\Big(\mathrm{d}p - q\mathrm{d}x\Big)^2 - 6\Big(\mathrm{d}z - 2q\mathrm{d}p + (q^2 - f - bz)\mathrm{d}x\Big)\mathrm{d}x -$$
$$2\Big(\mathrm{d}y - p\mathrm{d}x\Big)\Big(6\mathrm{d}q - 2b\mathrm{d}p - (\tfrac{2}{5}b^2 + \tfrac{9}{10}f_{pp})(\mathrm{d}y - p\mathrm{d}x) - (4bq + 3f_p)\mathrm{d}x\Big).$$

Ambient metrics for my favorite conformal structure

- Denoting by $\omega^1 = \mathrm{d}y - p\mathrm{d}x$ and by $\omega^4 = 3\mathrm{d}x$ I make an ansatz for the metric $g(x^i, \rho)$ which stays in the definition of $\tilde{g}$ by putting $g_\rho = \left(g_{\mathcal{D}} + A\cdot(\omega^1)^2 + 2B\cdot\omega^1\omega^4 + C\cdot(\omega^4)^2\right)$.
- That is to say that I look for an ambient metric in the form

$$\tilde{g}_{\mathcal{D}} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2\left(g_{\mathcal{D}} + A\cdot(\omega^1)^2 + 2B\cdot\omega^1\omega^4 + C\cdot(\omega^4)^2\right),$$

with *unknown* functions $A = A(x,p,\rho)$, $B = B(x,p,\rho)$ and $C = C(x,p,\rho)$.

Ambient metrics for my favorite conformal structure

- Denoting by $\omega^1 = \mathrm{d}y - p\mathrm{d}x$ and by $\omega^4 = 3\mathrm{d}x$ I make an ansatz for the metric $g(x^i, \rho)$ which stays in the definition of $\tilde{g}$ by putting $g_\rho = \Big(g_{\mathcal{D}} + A \cdot (\omega^1)^2 + 2B \cdot \omega^1\omega^4 + C \cdot (\omega^4)^2\Big)$.
- That is to say that I look for an ambient metric in the form

$$\tilde{g}_{\mathcal{D}} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2\Big(g_{\mathcal{D}} + A \cdot (\omega^1)^2 + 2B \cdot \omega^1\omega^4 + C \cdot (\omega^4)^2\Big),$$

with *unknown* functions $A = A(x, p, \rho)$, $B = B(x, p, \rho)$ and $C = C(x, p, \rho)$.

Theorem

The metric $\tilde{g}_{\mathcal{D}}$, as above, is an ambient metric for the conformal class $(M^5, [g_{\mathcal{D}_f}])$, if and only if the unknown functions $A = A(x, p, \rho)$, $B = B(x, p, \rho)$ and $C = C(x, p, \rho)$, satisfy the initial conditions $A_{|\rho=0} \equiv 0$, $B_{|\rho=0} \equiv 0$, $C_{|\rho=0} \equiv 0$ and the following system of PDEs:

$$LA = \frac{9}{40} f_{pppp}$$

$$LB = -\frac{1}{36} A_p + \frac{3}{40} f_{ppp}$$

$$LC = -\frac{1}{18} B_p + \frac{1}{324} A + \frac{1}{30} f_{pp} - \frac{2}{15} b^2,$$

with the *linear* operator L given by

$$L = 2\rho \frac{\partial^2}{\partial \rho^2} - 3 \frac{\partial}{\partial \rho} - \frac{1}{8} \frac{\partial^2}{\partial p^2}.$$

Theorem

The metric $\tilde{g}_{\mathcal{D}}$, as above, is an ambient metric for the conformal class $(M^5, [g_{\mathcal{D}_f}])$, if and only if the unknown functions $A = A(x, p, \rho)$, $B = B(x, p, \rho)$ and $C = C(x, p, \rho)$, satisfy the initial conditions $A_{|\rho=0} \equiv 0$, $B_{|\rho=0} \equiv 0$, $C_{|\rho=0} \equiv 0$ and the following system of PDEs:

$$LA = \frac{9}{40} f_{pppp}$$

$$LB = -\frac{1}{36} A_p + \frac{3}{40} f_{ppp}$$

$$LC = -\frac{1}{18} B_p + \frac{1}{324} A + \frac{1}{30} f_{pp} - \frac{2}{15} b^2,$$

with the *linear* operator L given by

$$L = 2\rho \frac{\partial^2}{\partial \rho^2} - 3 \frac{\partial}{\partial \rho} - \frac{1}{8} \frac{\partial^2}{\partial p^2}.$$

Theorem

The metric $\tilde{g}_{\mathcal{D}}$, as above, is an ambient metric for the conformal class $(M^5, [g_{\mathcal{D}_f}])$, if and only if the unknown functions $A = A(x,p,\rho)$, $B = B(x,p,\rho)$ and $C = C(x,p,\rho)$, satisfy the initial conditions $A_{|\rho=0} \equiv 0$, $B_{|\rho=0} \equiv 0$, $C_{|\rho=0} \equiv 0$ and the following system of PDEs:

$$LA = \tfrac{9}{40} f_{pppp}$$
$$LB = -\tfrac{1}{36} A_p + \tfrac{3}{40} f_{ppp}$$
$$LC = -\tfrac{1}{18} B_p + \tfrac{1}{324} A + \tfrac{1}{30} f_{pp} - \tfrac{2}{15} b^2,$$

with the *linear* operator L given by

$$L = 2\rho \frac{\partial^2}{\partial \rho^2} - 3\frac{\partial}{\partial \rho} - \tfrac{1}{8}\frac{\partial^2}{\partial p^2}.$$

Ambient metric for *pp*-waves

- A conformal manifold $(M^n, [g])$ contains a *pp*-wave metric g iff it admits coordinates (u, r, x^i), $i = 1, 2, \ldots, n-2$ in which g is given by $g = 2\mathrm{d}u(\mathrm{d}r + f\mathrm{d}u) + \sum_{i=1}^{n-2}(\mathrm{d}x^i)^2$. Here $f = f(x^i, u)$ is a differentiable function.
- Similar theorem: making an ansatz for the ambient metric in the form $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h\mathrm{d}u^2)$ with a differentiable function $h = h(x^i, u, \rho)$, one shows that the equations $Ric(\tilde{g}) = 0$ are equivalent to

$$Lh = \Delta f, \quad \text{with} \quad \Delta = \sum_{i=1}^{n-2} \frac{\partial^2}{(\partial x^i)^2},$$

and with the linear operator given by

$$L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho} - \Delta.$$

Ambient metric for *pp*-waves

- A conformal manifold $(M^n, [g])$ contains a *pp*-wave metric g iff it admits coordinates (u, r, x^i), $i = 1, 2, \ldots, n-2$ in which g is given by $g = 2\mathrm{d}u(\mathrm{d}r + f\mathrm{d}u) + \sum_{i=1}^{n-2}(\mathrm{d}x^i)^2$. Here $f = f(x^i, u)$ is a differentiable function.
- Similar theorem: making an ansatz for the ambient metric in the form $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h\mathrm{d}u^2)$ with a differentiable function $h = h(x^i, u, \rho)$, one shows that the equations $Ric(\tilde{g}) = 0$ are equivalent to

$$Lh = \Delta f, \quad \text{with} \quad \Delta = \sum_{i=1}^{n-2} \frac{\partial^2}{(\partial x^i)^2},$$

and with the linear operator given by

$$L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho} - \Delta.$$

Ambient metric for *pp*-waves

- A conformal manifold $(M^n, [g])$ contains a *pp*-wave metric g iff it admits coordinates (u, r, x^i), $i = 1, 2, \ldots, n-2$ in which g is given by $g = 2\mathrm{d}u(\mathrm{d}r + f\mathrm{d}u) + \sum_{i=1}^{n-2}(\mathrm{d}x^i)^2$. Here $f = f(x^i, u)$ is a differentiable function.
- Similar theorem: making an ansatz for the ambient metric in the form $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h\mathrm{d}u^2)$ with a differentiable function $h = h(x^i, u, \rho)$, one shows that the equations $Ric(\tilde{g}) = 0$ are equivalent to

$$Lh = \Delta f, \quad \text{with} \quad \Delta = \sum_{i=1}^{n-2} \frac{\partial^2}{(\partial x^i)^2},$$

and with the linear operator given by

$$L = 2\rho \frac{\partial^2}{\partial \rho^2} - (n-2)\frac{\partial}{\partial \rho} - \Delta.$$

Ambient metric for *pp*-waves

- A conformal manifold $(M^n, [g])$ contains a *pp*-wave metric g iff it admits coordinates (u, r, x^i), $i = 1, 2, \ldots, n-2$ in which g is given by $g = 2\mathrm{d}u(\mathrm{d}r + f\mathrm{d}u) + \sum_{i=1}^{n-2}(\mathrm{d}x^i)^2$. Here $f = f(x^i, u)$ is a differentiable function.
- Similar theorem: making an ansatz for the ambient metric in the form $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h\mathrm{d}u^2)$ with a differentiable function $h = h(x^i, u, \rho)$, one shows that the equations $Ric(\tilde{g}) = 0$ are equivalent to

$$Lh = \Delta f, \quad \text{with} \quad \Delta = \sum_{i=1}^{n-2} \frac{\partial^2}{(\partial x^i)^2},$$

and with the linear operator given by

$$L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho} - \Delta.$$

Ambient metric for *pp*-waves

- A conformal manifold $(M^n, [g])$ contains a *pp*-wave metric g iff it admits coordinates (u, r, x^i), $i = 1, 2, \ldots, n-2$ in which g is given by $g = 2\mathrm{d}u(\mathrm{d}r + f\mathrm{d}u) + \sum_{i=1}^{n-2}(\mathrm{d}x^i)^2$. Here $f = f(x^i, u)$ is a differentiable function.
- Similar theorem: making an ansatz for the ambient metric in the form $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h\mathrm{d}u^2)$ with a differentiable function $h = h(x^i, u, \rho)$, one shows that the equations $Ric(\tilde{g}) = 0$ are equivalent to

$$Lh = \Delta f, \quad \text{with} \quad \Delta = \sum_{i=1}^{n-2} \frac{\partial^2}{(\partial x^i)^2},$$

and with the linear operator given by

$$L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho} - \Delta.$$

Ambient metric for *pp*-waves

- A conformal manifold $(M^n, [g])$ contains a *pp*-wave metric g iff it admits coordinates (u, r, x^i), $i = 1, 2, \dots, n-2$ in which g is given by $g = 2\mathrm{d}u(\mathrm{d}r + f\mathrm{d}u) + \sum_{i=1}^{n-2}(\mathrm{d}x^i)^2$. Here $f = f(x^i, u)$ is a differentiable function.
- Similar theorem: making an ansatz for the ambient metric in the form $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h\mathrm{d}u^2)$ with a differentiable function $h = h(x^i, u, \rho)$, one shows that the equations $Ric(\tilde{g}) = 0$ are equivalent to

$$Lh = \Delta f, \quad \text{with} \quad \Delta = \sum_{i=1}^{n-2} \frac{\partial^2}{(\partial x^i)^2},$$

and with the linear operator given by

$$L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho} - \Delta.$$

Ambient metric for *pp*-waves

- A conformal manifold $(M^n, [g])$ contains a *pp*-wave metric g iff it admits coordinates (u, r, x^i), $i = 1, 2, \ldots, n-2$ in which g is given by $g = 2\mathrm{d}u(\mathrm{d}r + f\mathrm{d}u) + \sum_{i=1}^{n-2}(\mathrm{d}x^i)^2$. Here $f = f(x^i, u)$ is a differentiable function.
- Similar theorem: making an ansatz for the ambient metric in the form $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h\mathrm{d}u^2)$ with a differentiable function $h = h(x^i, u, \rho)$, one shows that the equations $Ric(\tilde{g}) = 0$ are equivalent to

$$Lh = \Delta f, \quad \text{with} \quad \Delta = \sum_{i=1}^{n-2} \frac{\partial^2}{(\partial x^i)^2},$$

and with the linear operator given by

$$L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho} - \Delta.$$

WARNING: Analytic solutions may not exist

- Let us look for solutions to $Lh = \Delta f$ in the form $h = \sum_{k=1}^{\infty} a_k(x^i, u)\rho^k$.
- Short calculation shows that

$$h = \sum_{k=1}^{\infty} \frac{\Delta^k f}{k! \prod_{j=1}^{k}(2j - n)} \rho^k.$$

- Since we have the product $\prod_{j=1}^{k}(2j - n)$ in the denominator, one sees that if the dimension n of the manifold M^n is even, $n = 2m$, then if $j = m$ the formula for h blows up. Thus *analytic* solutions exists only up to order $k = m - 1$, *unless* $\Delta^m f = 0$.
- This is the origin of the *Fefferman Graham obstruction*. The only $2m$-dimensional pp-waves that admit *analytic* solutions are those for which the defining function f satisfies the vanishing obstruction condition $\Delta^m f = 0$.

WARNING: Analytic solutions may not exist

- Let us look for solutions to $Lh = \Delta f$ in the form $h = \sum_{k=1}^{\infty} a_k(x^i, u)\rho^k$.
- Short calculation shows that

$$h = \sum_{k=1}^{\infty} \frac{\Delta^k f}{k! \prod_{j=1}^{k}(2j - n)} \rho^k.$$

- Since we have the product $\prod_{j=1}^{k}(2j - n)$ in the denominator, one sees that if the dimension n of the manifold M^n is even, $n = 2m$, then if $j = m$ the formula for h blows up. Thus *analytic* solutions exists only up to order $k = m - 1$, *unless* $\Delta^m f = 0$.
- This is the origin of the *Fefferman Graham obstruction*. The only $2m$-dimensional pp-waves that admit *analytic* solutions are those for which the defining function f satisfies the vanishing obstruction condition $\Delta^m f = 0$.

WARNING: Analytic solutions may not exist

- Let us look for solutions to $Lh = \Delta f$ in the form $h = \sum_{k=1}^{\infty} a_k(x^i, u)\rho^k$.
- Short calculation shows that

$$h = \sum_{k=1}^{\infty} \frac{\Delta^k f}{k! \prod_{j=1}^{k}(2j - n)} \rho^k.$$

- Since we have the product $\prod_{j=1}^{k}(2j - n)$ in the denominator, one sees that if the dimension n of the manifold M^n is even, $n = 2m$, then if $j = m$ the formula for h blows up. Thus *analytic* solutions exists only up to order $k = m - 1$, *unless* $\Delta^m f = 0$.
- This is the origin of the *Fefferman Graham obstruction*. The only $2m$-dimensional pp-waves that admit *analytic* solutions are those for which the defining function f satisfies the vanishing obstruction condition $\Delta^m f = 0$.

WARNING: Analytic solutions may not exist

- Let us look for solutions to $Lh = \Delta f$ in the form $h = \sum_{k=1}^{\infty} a_k(x^i, u)\rho^k$.
- Short calculation shows that

$$h = \sum_{k=1}^{\infty} \frac{\Delta^k f}{k! \prod_{j=1}^{k}(2j - n)} \rho^k.$$

- Since we have the product $\prod_{j=1}^{k}(2j - n)$ in the denominator, one sees that if the dimension n of the manifold M^n is even, $n = 2m$, then if $j = m$ the formula for h blows up. Thus *analytic* solutions exists only up to order $k = m - 1$, *unless* $\Delta^m f = 0$.
- This is the origin of the *Fefferman Graham obstruction*. The only $2m$-dimensional *pp*-waves that admit *analytic* solutions are those for which the defining function f satisfies the vanishing obstruction condition $\Delta^m f = 0$.

WARNING: Analytic solutions may not exist

- Let us look for solutions to $Lh = \Delta f$ in the form $h = \sum_{k=1}^{\infty} a_k(x^i, u)\rho^k$.
- Short calculation shows that
$$h = \sum_{k=1}^{\infty} \frac{\Delta^k f}{k! \prod_{j=1}^{k}(2j - n)} \rho^k.$$
- Since we have the product $\prod_{j=1}^{k}(2j - n)$ in the denominator, one sees that if the dimension n of the manifold M^n is even, $n = 2m$, then if $j = m$ the formula for h blows up. Thus *analytic* solutions exists only up to order $k = m - 1$, *unless* $\Delta^m f = 0$.
- This is the origin of the *Fefferman Graham obstruction*. The only $2m$-dimensional pp-waves that admit *analytic* solutions are those for which the defining function f satisfies the vanishing obstruction condition $\Delta^m f = 0$.

WARNING: Analytic solutions may not exist

- Let us look for solutions to $Lh = \Delta f$ in the form $h = \sum_{k=1}^{\infty} a_k(x^i, u)\rho^k$.
- Short calculation shows that

$$h = \sum_{k=1}^{\infty} \frac{\Delta^k f}{k! \prod_{j=1}^{k}(2j - n)} \rho^k.$$

- Since we have the product $\prod_{j=1}^{k}(2j - n)$ in the denominator, one sees that if the dimension n of the manifold M^n is even, $n = 2m$, then if $j = m$ the formula for h blows up. Thus *analytic* solutions exists only up to order $k = m - 1$, *unless* $\Delta^m f = 0$.
- This is the origin of the *Fefferman Graham obstruction*. The only $2m$-dimensional pp-waves that admit *analytic* solutions are those for which the defining function f satisfies the vanishing obstruction condition $\Delta^m f = 0$.

WARNING: Analytic solutions may not exist

- Let us look for solutions to $Lh = \Delta f$ in the form $h = \sum_{k=1}^{\infty} a_k(x^i, u)\rho^k$.
- Short calculation shows that

$$h = \sum_{k=1}^{\infty} \frac{\Delta^k f}{k! \prod_{j=1}^{k}(2j - n)} \rho^k.$$

- Since we have the product $\prod_{j=1}^{k}(2j - n)$ in the denominator, one sees that if the dimension n of the manifold M^n is even, $n = 2m$, then if $j = m$ the formula for h blows up. Thus *analytic* solutions exists only up to order $k = m - 1$, *unless* $\Delta^m f = 0$.
- This is the origin of the *Fefferman Graham obstruction*. The only $2m$-dimensional pp-waves that admit *analytic* solutions are those for which the defining function f satisfies the vanishing obstruction condition $\Delta^m f = 0$.

But there are nonanalytic ones

Theorem
When $n = 2m$ the most general solutions h with $h(\rho) \to 0$ when $\rho \downarrow 0$ are:

$$h = \rho^m\Big(\alpha + \sum_{k=1}^{\infty} \frac{\Delta^k \alpha}{k!\prod_{i=1}^{k}(2i+n)}\rho^k\Big) + \sum_{k=1}^{m-1} \frac{\Delta^k h}{k!\prod_{i=1}^{k}(2i-n)}\rho^k$$
$$+\ c_n\rho^m\left(\sum_{k=0}^{\infty}(\log(\rho) - q_k)\frac{\Delta^{m+k}h}{k!\prod_{i=1}^{k}(2i+n)}\rho^k\right) + c_n\rho^m\, Q * \sum_{k=0}^{\infty}\frac{\Delta^{m+k}h}{k!\prod_{i=1}^{k}(2i+n)}\rho^k,$$

where $\alpha = \alpha(x^i, u)$ and $Q = Q(x^i, u)$ are arbitrary functions of their variables, $*$ denotes the convolution of two functions with respect to the x^i-variables, and the constants are given as follows

$$c_n := -\frac{1}{(m-1)!\prod_{i=0}^{m-1}(2i-n)},\quad q_0 := 0,\quad q_k := \sum_{i=1}^{k}\frac{n+4i}{i(n+2i)},$$

for $k = 1, 2, \ldots$.
In particular, only when $\Delta^m h \equiv 0$ there are solutions that are analytic in ρ in a neighbourhood of $\rho = 0$ and with $h(0) = 0$.

But there are nonanalytic ones

Theorem

When $n = 2m$ the most general solutions h with $h(\rho) \to 0$ when $\rho \downarrow 0$ are:

$$h = \rho^m\Big(\alpha + \sum_{k=1}^{\infty} \frac{\Delta^k \alpha}{k! \prod_{i=1}^{k}(2i+n)}\rho^k\Big) + \sum_{k=1}^{m-1} \frac{\Delta^k h}{k! \prod_{i=1}^{k}(2i-n)}\rho^k$$

$$+ \; c_n \rho^m \left(\sum_{k=0}^{\infty} (\log(\rho) - q_k) \frac{\Delta^{m+k} h}{k! \prod_{i=1}^{k}(2i+n)}\rho^k \right) + c_n \rho^m \; Q * \sum_{k=0}^{\infty} \frac{\Delta^{m+k} h}{k! \prod_{i=1}^{k}(2i+n)}\rho^k,$$

where $\alpha = \alpha(x^i, u)$ and $Q = Q(x^i, u)$ are arbitrary functions of their variables, $*$ denotes the convolution of two functions with respect to the x^i-variables, and the constants are given as follows

$$c_n := -\frac{1}{(m-1)! \prod_{i=0}^{m-1}(2i-n)}, \; q_0 := 0, \; q_k := \sum_{i=1}^{k} \frac{n+4i}{i(n+2i)},$$

for $k = 1, 2, \ldots$.

In particular, only when $\Delta^m h \equiv 0$ there are solutions that are analytic in ρ in a neighbourhood of $\rho = 0$ and with $h(0) = 0$.

More examples

There are known more examples of conformal structures for which the Fefferman-Graham equations reduce to systems of linear PDEs. These include

- conformal classes of signature $(3,3)$ corresponding to special types of $(3,6)$ distributions,
- conformal Patterson-Walker metrics.

PROBLEM: explain what is the reason for such a phenomenon; or characterize those $[g]$ for which Fefferman-Graham equations reduce to linear PDEs.

More examples

There are known more examples of conformal structures for which the Fefferman-Graham equations reduce to systems of linear PDEs. These include

- conformal classes of signature $(3,3)$ corresponding to special types of $(3,6)$ distributions,
- conformal Patterson-Walker metrics.

PROBLEM: explain what is the reason for such a phenomenon; or characterize those $[g]$ for which Fefferman-Graham equations reduce to linear PDEs.

More examples

There are known more examples of conformal structures for which the Fefferman-Graham equations reduce to systems of linear PDEs. These include

- conformal classes of signature $(3,3)$ corresponding to special types of $(3,6)$ distributions,
- conformal Patterson-Walker metrics.

PROBLEM: explain what is the reason for such a phenomenon; or characterize those $[g]$ for which Fefferman-Graham equations reduce to linear PDEs.

More examples

There are known more examples of conformal structures for which the Fefferman-Graham equations reduce to systems of linear PDEs. These include

- conformal classes of signature $(3,3)$ corresponding to special types of $(3,6)$ distributions,
- conformal Patterson-Walker metrics.

PROBLEM: explain what is the reason for such a phenomenon; or characterize those $[g]$ for which Fefferman-Graham equations reduce to linear PDEs.

More examples

There are known more examples of conformal structures for which the Fefferman-Graham equations reduce to systems of linear PDEs. These include

- conformal classes of signature $(3,3)$ corresponding to special types of $(3,6)$ distributions,
- conformal Patterson-Walker metrics.

PROBLEM: explain what is the reason for such a phenomenon; or characterize those $[g]$ for which Fefferman-Graham equations reduce to linear PDEs.

More examples

There are known more examples of conformal structures for which the Fefferman-Graham equations reduce to systems of linear PDEs. These include

- conformal classes of signature $(3, 3)$ corresponding to special types of $(3, 6)$ distributions,
- conformal Patterson-Walker metrics.

PROBLEM: explain what is the reason for such a phenomenon; or characterize those $[g]$ for which Fefferman-Graham equations reduce to linear PDEs.

More examples

There are known more examples of conformal structures for which the Fefferman-Graham equations reduce to systems of linear PDEs. These include

- conformal classes of signature $(3, 3)$ corresponding to special types of $(3, 6)$ distributions,
- conformal Patterson-Walker metrics.

PROBLEM: explain what is the reason for such a phenomenon; or characterize those $[g]$ for which Fefferman-Graham equations reduce to linear PDEs.

More examples

There are known more examples of conformal structures for which the Fefferman-Graham equations reduce to systems of linear PDEs. These include

- conformal classes of signature $(3,3)$ corresponding to special types of $(3,6)$ distributions,
- conformal Patterson-Walker metrics.

PROBLEM: explain what is the reason for such a phenomenon; or characterize those $[g]$ for which Fefferman-Graham equations reduce to linear PDEs.

More examples

There are known more examples of conformal structures for which the Fefferman-Graham equations reduce to systems of linear PDEs. These include

- conformal classes of signature $(3,3)$ corresponding to special types of $(3,6)$ distributions,
- conformal Patterson-Walker metrics.

PROBLEM: explain what is the reason for such a phenomenon; or characterize those $[g]$ for which Fefferman-Graham equations reduce to linear PDEs.

Plan

Perturbation h and a crucial observation

- We take g from $[g]$, where $(M^n, [g])$ is any conformal structure, and write the Fefferman-Graham metric as:

$$\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h),$$

  where $h = h(x^i, \rho)$ and $h(x^i, 0) = 0$.
- The goal is to rewrite the Fefferman-Graham equations in terms of $h = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ rather than in terms of $g(x^k, \rho)_{ij}$.
- This would be a pain for general h but we additionally assume that $h^i{}_j = g^{ik}h_{kj}$ is *2-step nilpotent*, $h^i{}_j h^j{}_k = 0$. This is equivalent to the assumption that

$$Im(h^i{}_j) \subset \mathcal{N},$$

  with $\mathcal{N}$ being *totally null* with respect to g.

Perturbation h and a crucial observation

- We take g from $[g]$, where $(M^n, [g])$ is any conformal structure, and write the Fefferman-Graham metric as:

$$\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h),$$

where $h = h(x^i, \rho)$ and $h(x^i, 0) = 0$.

- The goal is to rewrite the Fefferman-Graham equations in terms of $h = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ rather than in terms of $g(x^k, \rho)_{ij}$.
- This would be a pain for general h but we additionally assume that $h^i{}_j = g^{ik}h_{kj}$ is *2-step nilpotent*, $h^i{}_j h^j{}_k = 0$. This is equivalent to the assumption that

$$Im(h^i{}_j) \subset \mathcal{N},$$

with $\mathcal{N}$ being *totally null* with respect to g.

Perturbation h and a crucial observation

- We take g from $[g]$, where $(M^n, [g])$ is any conformal structure, and write the Fefferman-Graham metric as:

$$\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h),$$

where $h = h(x^i, \rho)$ and $h(x^i, 0) = 0$.

- The goal is to rewrite the Fefferman-Graham equations in terms of $h = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ rather than in terms of $g(x^k, \rho)_{ij}$.
- This would be a pain for general h but we additionally assume that $h^i{}_j = g^{ik}h_{kj}$ is *2-step nilpotent*, $h^i{}_j h^j{}_k = 0$. This is equivalent to the assumption that

$$Im(h^i{}_j) \subset \mathcal{N},$$

with $\mathcal{N}$ being *totally null* with respect to g.

Perturbation h and a crucial observation

- We take g from $[g]$, where $(M^n, [g])$ is any conformal structure, and write the Fefferman-Graham metric as:

$$\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h),$$

  where $h = h(x^i, \rho)$ and $h(x^i, 0) = 0$.
- The goal is to rewrite the Fefferman-Graham equations in terms of $h = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ rather than in terms of $g(x^k, \rho)_{ij}$.
- This would be a pain for general h but we additionally assume that $h^i{}_j = g^{ik}h_{kj}$ is *2-step nilpotent*, $h^i{}_j h^j{}_k = 0$. This is equivalent to the assumption that

$$Im(h^i{}_j) \subset \mathcal{N},$$

  with $\mathcal{N}$ being *totally null* with respect to g.

Perturbation h and a crucial observation

- We take g from $[g]$, where $(M^n, [g])$ is any conformal structure, and write the Fefferman-Graham metric as:

$$\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h),$$

  where $h = h(x^i, \rho)$ and $h(x^i, 0) = 0$.
- The goal is to rewrite the Fefferman-Graham equations in terms of $h = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ rather than in terms of $g(x^k, \rho)_{ij}$.
- This would be a pain for general h but we additionally assume that $h^i{}_j = g^{ik}h_{kj}$ is *2-step nilpotent*, $h^i{}_j h^j{}_k = 0$. This is equivalent to the assumption that

$$Im(h^i{}_j) \subset \mathcal{N},$$

  with $\mathcal{N}$ being *totally null* with respect to g.

Perturbation h and a crucial observation

- The assumption about 2-step nilpotency of h solves the problem of getting inverses $g(x^k, \rho)^{ij}$ of $g(x^k, \rho)_{ij}$.
- We have $g(x^k, \rho)_{ij} = g_{ij} + h_{ij}$, and because of the 2-step nilpotency of h, we get

$$g(x^k, \rho)^{ij} = g^{ij} - h^{ij}.$$

- We note that *all our examples of structures with linear Fefferman-Graham equations presented here have h which are 2-step nilpotent*.
- We also note that since $Im(h) \subset \mathcal{N}$, then $Tr(h) = 0$.

Perturbation h and a crucial observation

- The assumption about 2-step nilpotency of h solves the problem of getting inverses $g(x^k, \rho)^{ij}$ of $g(x^k, \rho)_{ij}$.
- We have $g(x^k, \rho)_{ij} = g_{ij} + h_{ij}$, and because of the 2-step nilpotency of h, we get

$$g(x^k, \rho)^{ij} = g^{ij} - h^{ij}.$$

- We note that *all our examples of structures with linear Fefferman-Graham equations presented here have h which are 2-step nilpotent*.
- We also note that since $Im(h) \subset \mathcal{N}$, then $Tr(h) = 0$.

Perturbation h and a crucial observation

- The assumption about 2-step nilpotency of h solves the problem of getting inverses $g(x^k, \rho)^{ij}$ of $g(x^k, \rho)_{ij}$.
- We have $g(x^k, \rho)_{ij} = g_{ij} + h_{ij}$, and because of the 2-step nilpotency of h, we get

$$g(x^k, \rho)^{ij} = g^{ij} - h^{ij}.$$

- We note that *all our examples of structures with linear Fefferman-Graham equations presented here have h which are 2-step nilpotent*.
- We also note that since $Im(h) \subset \mathcal{N}$, then $Tr(h) = 0$.

Perturbation h and a crucial observation

- The assumption about 2-step nilpotency of h solves the problem of getting inverses $g(x^k,\rho)^{ij}$ of $g(x^k,\rho)_{ij}$.
- We have $g(x^k,\rho)_{ij} = g_{ij} + h_{ij}$, and because of the 2-step nilpotency of h, we get

$$g(x^k,\rho)^{ij} = g^{ij} - h^{ij}.$$

- We note that *all our examples of structures with linear Fefferman-Graham equations presented here have h which are 2-step nilpotent*.
- We also note that since $Im(h) \subset \mathcal{N}$, then $Tr(h) = 0$.

Perturbation h and a crucial observation

- The assumption about 2-step nilpotency of h solves the problem of getting inverses $g(x^k, \rho)^{ij}$ of $g(x^k, \rho)_{ij}$.
- We have $g(x^k, \rho)_{ij} = g_{ij} + h_{ij}$, and because of the 2-step nilpotency of h, we get

$$g(x^k, \rho)^{ij} = g^{ij} - h^{ij}.$$

- We note that *all our examples of structures with linear Fefferman-Graham equations presented here have h which are 2-step nilpotent*.
- We also note that since $Im(h) \subset \mathcal{N}$, then $Tr(h) = 0$.

Fefferman-Graham equations in terms of h s.t. $h^2 = 0$

Proposition

Assume that $Im(h) \subset \mathcal{N}$, with $\mathcal{N}$ totally null. Then the equations $Ric(\tilde{g}) = 0$ for $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ are equivalent to the following two sets of PDEs:

$$g^{kl}\nabla_k \dot{h}_{il} = 0,$$

$$Lh_{ij} + 2\nabla^k\nabla_{(i}h_{j)k} - \nabla^k\nabla_k h_{ij} + 2R_{ij} + Q^{(2)}{}_{ij} + Q^{(3)}{}_{ij} + Q^{(4)}_{ij} = 0.$$

Here L is the linear operator

$$L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho},$$

R_{ij} is the Ricci tensor of g and the red terms $Q^{(i)}$, $i = 2, 3, 4$ depend on h quadratically (i=2), cubically (i=3) and quartically (i=4).

Fefferman-Graham equations in terms of h s.t. $h^2 = 0$

Proposition

Assume that $Im(h) \subset \mathcal{N}$, with $\mathcal{N}$ totally null. Then the equations $Ric(\tilde{g}) = 0$ for $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ are equivalent to the following two sets of PDEs:

$$g^{kl}\nabla_k \dot{h}_{il} = 0,$$

$$Lh_{ij} + 2\nabla^k\nabla_{(i}h_{j)k} - \nabla^k\nabla_k h_{ij} + 2R_{ij} + Q^{(2)}{}_{ij} + Q^{(3)}{}_{ij} + Q^{(4)}_{ij} = 0.$$

Here L is the linear operator

$$L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho},$$

R_{ij} is the Ricci tensor of g and the red terms $Q^{(i)}$, $i = 2, 3, 4$ depend on h quadratically (i=2), cubically (i=3) and quartically (i=4).

Fefferman-Graham equations in terms of h s.t. $h^2 = 0$

Proposition

Assume that $Im(h) \subset \mathcal{N}$, with $\mathcal{N}$ totally null. Then the equations $Ric(\tilde{g}) = 0$ for $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ are equivalent to the following two sets of PDEs:

$$g^{kl}\nabla_k \dot{h}_{il} = 0,$$

$$Lh_{ij} + 2\nabla^k\nabla_{(i}h_{j)k} - \nabla^k\nabla_k h_{ij} + 2R_{ij} + Q^{(2)}{}_{ij} + Q^{(3)}{}_{ij} + Q^{(4)}_{ij} = 0.$$

Here L is the linear operator

$$L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho},$$

R_{ij} is the Ricci tensor of g and the red terms $Q^{(i)}$, $i = 2, 3, 4$ depend on h quadratically (i=2), cubically (i=3) and quartically (i=4).

Fefferman-Graham equations in terms of h s.t. $h^2 = 0$

Proposition

Assume that $Im(h) \subset \mathcal{N}$, with $\mathcal{N}$ totally null. Then the equations $Ric(\tilde{g}) = 0$ for $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ are equivalent to the following two sets of PDEs:

$$g^{kl}\nabla_k \dot{h}_{il} = 0,$$

$$Lh_{ij} + 2\nabla^k\nabla_{(i}h_{j)k} - \nabla^k\nabla_k h_{ij} + 2R_{ij} + Q^{(2)}{}_{ij} + Q^{(3)}{}_{ij} + Q^{(4)}_{ij} = 0.$$

Here L is the linear operator

$$L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho},$$

R_{ij} is the Ricci tensor of g and the red terms $Q^{(i)}$, $i = 2, 3, 4$ depend on h quadratically (i=2), cubically (i=3) and quartically (i=4).

Fefferman-Graham equations in terms of h s.t. $h^2 = 0$

Proposition

Assume that $Im(h) \subset \mathcal{N}$, with $\mathcal{N}$ totally null. Then the equations $Ric(\tilde{g}) = 0$ for $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ are equivalent to the following two sets of PDEs:

$$g^{kl}\nabla_k \dot{h}_{il} = 0,$$

$$Lh_{ij} + 2\nabla^k\nabla_{(i}h_{j)k} - \nabla^k\nabla_k h_{ij} + 2R_{ij} + Q^{(2)}{}_{ij} + Q^{(3)}{}_{ij} + Q^{(4)}_{ij} = 0.$$

Here L is the linear operator

$$L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho},$$

R_{ij} is the Ricci tensor of g and the red terms $Q^{(i)}$, $i = 2, 3, 4$ depend on h quadratically (i=2), cubically (i=3) and quartically (i=4).

Fefferman-Graham equations in terms of h s.t. $h^2=0$

Proposition

Assume that $Im(h) \subset \mathcal{N}$, with $\mathcal{N}$ totally null. Then the equations $Ric(\tilde{g}) = 0$ for $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g+h)$ are equivalent to the following two sets of PDEs:

$$g^{kl}\nabla_k \dot{h}_{il} = 0,$$

$$Lh_{ij} + 2\nabla^k\nabla_{(i}h_{j)k} - \nabla^k\nabla_k h_{ij} + 2R_{ij} + {Q^{(2)}}_{ij} + {Q^{(3)}}_{ij} + Q^{(4)}_{ij} = 0.$$

Here L is the linear operator

$$L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho},$$

R_{ij} is the Ricci tensor of g and the red terms $Q^{(i)}$, $i=2,3,4$ depend on h quadratically (i=2), cubically (i=3) and quartically (i=4).

Plan

1. Ambient metrics and distributions
   - Fefferman-Graham construction
   - Ambient metrics for special conformal structures
2. Fefferman-Graham equations in terms of a perturbation h
   - Passing from g_ρ to $g + h_\rho$
3. Results
   - Theorems
   - New examples

First simplification

Proposition
Assume that $Im(h) \subset \mathcal{N}$, with $\mathcal{N}$ totally null, and that

$$\mathcal{N}^{\perp} \quad \text{is integrable.}$$

Then the equations $Ric(\tilde{g}) = 0$ for $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ are equivalent to the following two sets of PDEs:

$$g^{kl}\nabla_k \dot{h}_{il} = 0,$$

$$Lh_{ij} + 2\nabla^k\nabla_{(i}h_{j)k} - \nabla^k\nabla_k h_{ij} + 2R_{ij} + Q^{(2)}{}_{ij} = 0.$$

Here L, R_{ij} and $Q^{(2)}$ are as before.

First simplification

Proposition

Assume that $Im(h) \subset \mathcal{N}$, with $\mathcal{N}$ totally null, and that

$$\mathcal{N}^{\perp} \quad \text{is integrable.}$$

Then the equations $Ric(\tilde{g}) = 0$ for $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ are equivalent to the following two sets of PDEs:

$$g^{kl}\nabla_k \dot{h}_{il} = 0,$$

$$Lh_{ij} + 2\nabla^k\nabla_{(i}h_{j)k} - \nabla^k\nabla_k h_{ij} + 2R_{ij} + Q^{(2)}{}_{ij} = 0.$$

Here L, R_{ij} and $Q^{(2)}$ are as before.

First simplification

Proposition

Assume that $Im(h) \subset \mathcal{N}$, with $\mathcal{N}$ totally null, and that

$$\mathcal{N}^\perp \quad \text{is integrable.}$$

Then the equations $Ric(\tilde{g}) = 0$ for $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ are equivalent to the following two sets of PDEs:

$$g^{kl}\nabla_k \dot{h}_{il} = 0,$$

$$Lh_{ij} + 2\nabla^k\nabla_{(i}h_{j)k} - \nabla^k\nabla_k h_{ij} + 2R_{ij} + Q^{(2)}{}_{ij} = 0.$$

Here L, R_{ij} and $Q^{(2)}$ are as before.

First simplification

Proposition
Assume that $Im(h) \subset \mathcal{N}$, with $\mathcal{N}$ totally null, and that

$$\mathcal{N}^{\perp} \quad \text{is integrable.}$$

Then the equations $Ric(\tilde{g}) = 0$ for $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ are equivalent to the following two sets of PDEs:

$$g^{kl}\nabla_k \dot{h}_{il} = 0,$$

$$Lh_{ij} + 2\nabla^k\nabla_{(i}h_{j)k} - \nabla^k\nabla_k h_{ij} + 2R_{ij} + {Q^{(2)}}_{ij} = 0.$$

Here L, R_{ij} and $Q^{(2)}$ are as before.

First simplification

Proposition

Assume that $Im(h) \subset \mathcal{N}$, with $\mathcal{N}$ totally null, and that

$$\mathcal{N}^{\perp} \quad \text{is integrable.}$$

Then the equations $Ric(\tilde{g}) = 0$ for $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ are equivalent to the following two sets of PDEs:

$$g^{kl}\nabla_k \dot{h}_{il} = 0,$$

$$Lh_{ij} + 2\nabla^k\nabla_{(i}h_{j)k} - \nabla^k\nabla_k h_{ij} + 2R_{ij} + Q^{(2)}{}_{ij} = 0.$$

Here L, R_{ij} and $Q^{(2)}$ are as before.

First simplification

Proposition
Assume that $Im(h) \subset \mathcal{N}$, with $\mathcal{N}$ totally null, and that

$$\mathcal{N}^{\perp} \quad \text{is integrable.}$$

Then the equations $Ric(\tilde{g}) = 0$ for $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g+h)$ are equivalent to the following two sets of PDEs:

$$g^{kl}\nabla_k \dot{h}_{il} = 0,$$

$$Lh_{ij} + 2\nabla^k\nabla_{(i}h_{j)k} - \nabla^k\nabla_k h_{ij} + 2R_{ij} + Q^{(2)}{}_{ij} = 0.$$

Here L, R_{ij} and $Q^{(2)}$ are as before.

First simplification

Proposition

Assume that $Im(h) \subset \mathcal{N}$, with $\mathcal{N}$ totally null, and that

$$\mathcal{N}^{\perp} \quad \text{is integrable.}$$

Then the equations $Ric(\tilde{g}) = 0$ for $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g+h)$ are equivalent to the following two sets of PDEs:

$$g^{kl}\nabla_k \dot{h}_{il} = 0,$$

$$Lh_{ij} + 2\nabla^k\nabla_{(i}h_{j)k} - \nabla^k\nabla_k h_{ij} + 2R_{ij} + Q^{(2)}{}_{ij} = 0.$$

Here L, R_{ij} and $Q^{(2)}$ are as before.

First simplification

Proposition

Assume that $Im(h) \subset \mathcal{N}$, with $\mathcal{N}$ totally null, and that

$$\mathcal{N}^{\perp} \quad \text{is integrable.}$$

Then the equations $Ric(\tilde{g}) = 0$ for $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ are equivalent to the following two sets of PDEs:

$$g^{kl}\nabla_k \dot{h}_{il} = 0,$$

$$Lh_{ij} + 2\nabla^k\nabla_{(i}h_{j)k} - \nabla^k\nabla_k h_{ij} + 2R_{ij} + Q^{(2)}{}_{ij} = 0.$$

Here L, R_{ij} and $Q^{(2)}$ are as before.

The best theorem we have

Theorem

Let g be a representative of a conformal class $[g]$ on a manifold M^n, let $\mathcal{N}$ be a totally null distribution on M^n, and $h = h(x^i, \rho)$ be a 1-parameter family of bilinear forms on M^n such that:

1. $Im(h) \subset \mathcal{N}$ with $\mathcal{N}^\perp$ integrable,
2. $\nabla_Z Y \subset \mathcal{N}, \quad \forall Y, Z \in \mathcal{N}$,
3. $\nabla_X Y \subset \mathcal{N}^\perp, \quad Y \in \mathcal{N}, X \in \mathrm{T}M^n$,
4. $\mathcal{L}_Y h = 0, \quad \forall Y \in \mathcal{N}$.

Then all the Fefferman-Graham equations $Ric(\tilde{g}) = 0$ for $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ are *linear* in h. Explicitly:

$$g^{kl}\nabla_k \dot{h}_{il} = 0,$$

$$Lh_{ij} + 2\nabla^k\nabla_{(i}h_{j)k} - \nabla^k\nabla_k h_{ij} + 2R_{ij} = 0.$$

with $L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho}$.

The best theorem we have

Theorem

Let g be a representative of a conformal class $[g]$ on a manifold M^n, let $\mathcal{N}$ be a totally null distribution on M^n, and $h = h(x^i, \rho)$ be a 1-parameter family of bilinear forms on M^n such that:

1. $Im(h) \subset \mathcal{N}$ with $\mathcal{N}^\perp$ integrable,
2. $\nabla_Z Y \subset \mathcal{N}, \quad \forall Y, Z \in \mathcal{N}$,
3. $\nabla_X Y \subset \mathcal{N}^\perp, \quad Y \in \mathcal{N}, X \in \mathrm{T}M^n$,
4. $\mathcal{L}_Y h = 0, \quad \forall Y \in \mathcal{N}$.

Then all the Fefferman-Graham equations $Ric(\tilde{g}) = 0$ for $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ are *linear* in h. Explicitly:

$$g^{kl}\nabla_k \dot{h}_{il} = 0,$$

$$Lh_{ij} + 2\nabla^k\nabla_{(i}h_{j)k} - \nabla^k\nabla_k h_{ij} + 2R_{ij} = 0.$$

with $L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho}$.

The best theorem we have

Theorem

Let g be a representative of a conformal class $[g]$ on a manifold M^n, let $\mathcal{N}$ be a totally null distribution on M^n, and $h = h(x^i, \rho)$ be a 1-parameter family of bilinear forms on M^n such that:

1. $Im(h) \subset \mathcal{N}$ with $\mathcal{N}^\perp$ integrable,
2. $\nabla_Z Y \subset \mathcal{N}, \quad \forall Y, Z \in \mathcal{N}$,
3. $\nabla_X Y \subset \mathcal{N}^\perp, \quad Y \in \mathcal{N}, X \in \mathrm{T}M^n$,
4. $\mathcal{L}_Y h = 0, \quad \forall Y \in \mathcal{N}$.

Then all the Fefferman-Graham equations $Ric(\tilde{g}) = 0$ for $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ are *linear* in h. Explicitly:

$$g^{kl}\nabla_k \dot{h}_{il} = 0,$$

$$Lh_{ij} + 2\nabla^k\nabla_{(i}h_{j)k} - \nabla^k\nabla_k h_{ij} + 2R_{ij} = 0.$$

with $L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho}$.

Null Ricci Walker metrics

Let $\mathbf{P}$ be a Schouten tensor for a metric g on M^n and let $\mathcal{N}$ be a totally null distribution on M^n. Consider two conditions

(A) $Im(\mathbf{P}) \subset \mathcal{N}$,

(B) $\mathcal{N}$ is parallel, i.e. $\nabla_X \mathcal{N} \subset \mathcal{N}$ for all $X \in \mathrm{T}M^n$.

If g satisfies condition (A) it is called *null Ricci*. If g satisfies condition (B) it is called *Walker*. Metrics g satisfying both conditions (A) and (B) are called *null Ricci Walker*.

Fact

Every Walker metric (hence every null Ricci Walker metric) with its defining totally null plane $\mathcal{N}$ has *integrable* $\mathcal{N}^\perp$.

QUESTION: What if I restrict to null Ricci Walker metrics and take h such that its image is in the same $\mathcal{N}$ as image of $\mathbf{P}$?

IMMEDIATE ANSWER: Fefferman-Graham equations will be at most *quadratic* in h. But...maybe more...

Null Ricci Walker metrics

Let $\mathbf{P}$ be a Schouten tensor for a metric g on M^n and let $\mathcal{N}$ be a totally null distribution on M^n. Consider two conditions

(A) $Im(\mathbf{P}) \subset \mathcal{N}$,

(B) $\mathcal{N}$ is parallel, i.e. $\nabla_X \mathcal{N} \subset \mathcal{N}$ for all $X \in \mathrm{T}M^n$.

If g satisfies condition (A) it is called *null Ricci*. If g satisfies condition (B) it is called *Walker*. Metrics g satisfying both conditions (A) and (B) are called *null Ricci Walker*.

Fact

Every Walker metric (hence every null Ricci Walker metric) with its defining totally null plane $\mathcal{N}$ has *integrable* $\mathcal{N}^\perp$.

QUESTION: What if I restrict to null Ricci Walker metrics and take h such that its image is in the same $\mathcal{N}$ as image of $\mathbf{P}$?

IMMEDIATE ANSWER: Fefferman-Graham equations will be at most *quadratic* in h. But...maybe more...

Null Ricci Walker metrics

Let $\mathbf{P}$ be a Schouten tensor for a metric g on M^n and let $\mathcal{N}$ be a totally null distribution on M^n. Consider two conditions

(A) $Im(\mathbf{P}) \subset \mathcal{N}$,

(B) $\mathcal{N}$ is parallel, i.e. $\nabla_X \mathcal{N} \subset \mathcal{N}$ for all $X \in \mathrm{T}M^n$.

If g satisfies condition (A) it is called *null Ricci*. If g satisfies condition (B) it is called *Walker*. Metrics g satisfying both conditions (A) and (B) are called *null Ricci Walker*.

Fact

Every Walker metric (hence every null Ricci Walker metric) with its defining totally null plane $\mathcal{N}$ has *integrable* $\mathcal{N}^\perp$.

QUESTION: What if I restrict to null Ricci Walker metrics and take h such that its image is in the same $\mathcal{N}$ as image of $\mathbf{P}$?

IMMEDIATE ANSWER: Fefferman-Graham equations will be at most *quadratic* in h. But...maybe more...

Null Ricci Walker metrics

Let $\mathbf{P}$ be a Schouten tensor for a metric g on M^n and let $\mathcal{N}$ be a totally null distribution on M^n. Consider two conditions

(A) $Im(\mathbf{P}) \subset \mathcal{N}$,

(B) $\mathcal{N}$ is parallel, i.e. $\nabla_X \mathcal{N} \subset \mathcal{N}$ for all $X \in \mathrm{T}M^n$.

If g satisfies condition (A) it is called *null Ricci*. If g satisfies condition (B) it is called *Walker*. Metrics g satisfying both conditions (A) and (B) are called *null Ricci Walker*.

Fact

Every Walker metric (hence every null Ricci Walker metric) with its defining totally null plane $\mathcal{N}$ has *integrable* $\mathcal{N}^\perp$.

QUESTION: What if I restrict to null Ricci Walker metrics and take h such that its image is in the same $\mathcal{N}$ as image of $\mathbf{P}$?

IMMEDIATE ANSWER: Fefferman-Graham equations will be at most *quadratic* in h. But...maybe more...

Null Ricci Walker metrics

Let $\mathbf{P}$ be a Schouten tensor for a metric g on M^n and let $\mathcal{N}$ be a totally null distribution on M^n. Consider two conditions

(A) $Im(\mathbf{P}) \subset \mathcal{N}$,

(B) $\mathcal{N}$ is parallel, i.e. $\nabla_X \mathcal{N} \subset \mathcal{N}$ for all $X \in \mathrm{T}M^n$.

If g satisfies condition (A) it is called *null Ricci*. If g satisfies condition (B) it is called *Walker*. Metrics g satisfying both conditions (A) and (B) are called *null Ricci Walker*.

Fact

Every Walker metric (hence every null Ricci Walker metric) with its defining totally null plane $\mathcal{N}$ has *integrable* $\mathcal{N}^\perp$.

QUESTION: What if I restrict to null Ricci Walker metrics and take h such that its image is in the same $\mathcal{N}$ as image of $\mathbf{P}$?

IMMEDIATE ANSWER: Fefferman-Graham equations will be at most *quadratic* in h. But...maybe more...

Null Ricci Walker metrics

Let $\mathbf{P}$ be a Schouten tensor for a metric g on M^n and let $\mathcal{N}$ be a totally null distribution on M^n. Consider two conditions

(A) $Im(\mathbf{P}) \subset \mathcal{N}$,

(B) $\mathcal{N}$ is parallel, i.e. $\nabla_X \mathcal{N} \subset \mathcal{N}$ for all $X \in \mathrm{T}M^n$.

If g satisfies condition (A) it is called *null Ricci*. If g satisfies condition (B) it is called *Walker*. Metrics g satisfying both conditions (A) and (B) are called *null Ricci Walker*.

Fact

Every Walker metric (hence every null Ricci Walker metric) with its defining totally null plane $\mathcal{N}$ has *integrable* $\mathcal{N}^\perp$.

QUESTION: What if I restrict to null Ricci Walker metrics and take h such that its image is in the same $\mathcal{N}$ as image of $\mathbf{P}$?

IMMEDIATE ANSWER: Fefferman-Graham equations will be at most *quadratic* in h. But...maybe more...

Null Ricci Walker metrics

Let $\mathbf{P}$ be a Schouten tensor for a metric g on M^n and let $\mathcal{N}$ be a totally null distribution on M^n. Consider two conditions

(A) $Im(\mathbf{P}) \subset \mathcal{N}$,

(B) $\mathcal{N}$ is parallel, i.e. $\nabla_X \mathcal{N} \subset \mathcal{N}$ for all $X \in \mathrm{T}M^n$.

If g satisfies condition (A) it is called *null Ricci*. If g satisfies condition (B) it is called *Walker*. Metrics g satisfying both conditions (A) and (B) are called *null Ricci Walker*.

Fact

Every Walker metric (hence every null Ricci Walker metric) with its defining totally null plane $\mathcal{N}$ has *integrable* $\mathcal{N}^\perp$.

QUESTION: What if I restrict to null Ricci Walker metrics and take h such that its image is in the same $\mathcal{N}$ as image of $\mathbf{P}$?

IMMEDIATE ANSWER: Fefferman-Graham equations will be at most *quadratic* in h. But...maybe more...

Null Ricci Walker metrics

Let $\mathbf{P}$ be a Schouten tensor for a metric g on M^n and let $\mathcal{N}$ be a totally null distribution on M^n. Consider two conditions

(A) $Im(\mathbf{P}) \subset \mathcal{N}$,

(B) $\mathcal{N}$ is parallel, i.e. $\nabla_X \mathcal{N} \subset \mathcal{N}$ for all $X \in \mathrm{T}M^n$.

If g satisfies condition (A) it is called *null Ricci*. If g satisfies condition (B) it is called *Walker*. Metrics g satisfying both conditions (A) and (B) are called *null Ricci Walker*.

Fact

Every Walker metric (hence every null Ricci Walker metric) with its defining totally null plane $\mathcal{N}$ has *integrable* $\mathcal{N}^\perp$.

QUESTION: What if I restrict to null Ricci Walker metrics and take h such that its image is in the same $\mathcal{N}$ as image of $\mathbf{P}$?

IMMEDIATE ANSWER: Fefferman-Graham equations will be at most *quadratic* in h. But...maybe more...

Null Ricci Walker metrics

Let $\mathbf{P}$ be a Schouten tensor for a metric g on M^n and let $\mathcal{N}$ be a totally null distribution on M^n. Consider two conditions

(A) $Im(\mathbf{P}) \subset \mathcal{N}$,

(B) $\mathcal{N}$ is parallel, i.e. $\nabla_X \mathcal{N} \subset \mathcal{N}$ for all $X \in \mathrm{T}M^n$.

If g satisfies condition (A) it is called *null Ricci*. If g satisfies condition (B) it is called *Walker*. Metrics g satisfying both conditions (A) and (B) are called *null Ricci Walker*.

Fact

Every Walker metric (hence every null Ricci Walker metric) with its defining totally null plane $\mathcal{N}$ has *integrable* $\mathcal{N}^\perp$.

QUESTION: What if I restrict to null Ricci Walker metrics and take h such that its image is in the same $\mathcal{N}$ as image of $\mathbf{P}$?

IMMEDIATE ANSWER: Fefferman-Graham equations will be at most *quadratic* in h. But...maybe more...

Null Ricci Walker metrics

Let $\mathbf{P}$ be a Schouten tensor for a metric g on M^n and let $\mathcal{N}$ be a totally null distribution on M^n. Consider two conditions

(A) $Im(\mathbf{P}) \subset \mathcal{N}$,

(B) $\mathcal{N}$ is parallel, i.e. $\nabla_X \mathcal{N} \subset \mathcal{N}$ for all $X \in \mathrm{T}M^n$.

If g satisfies condition (A) it is called *null Ricci*. If g satisfies condition (B) it is called *Walker*. Metrics g satisfying both conditions (A) and (B) are called *null Ricci Walker*.

Fact

Every Walker metric (hence every null Ricci Walker metric) with its defining totally null plane $\mathcal{N}$ has *integrable* $\mathcal{N}^\perp$.

QUESTION: What if I restrict to null Ricci Walker metrics and take h such that its image is in the same $\mathcal{N}$ as image of $\mathbf{P}$?

IMMEDIATE ANSWER: Fefferman-Graham equations will be at most *quadratic* in h. But...maybe more...

Null Ricci Walker metrics

Let $\mathbf{P}$ be a Schouten tensor for a metric g on M^n and let $\mathcal{N}$ be a totally null distribution on M^n. Consider two conditions

(A) $Im(\mathbf{P}) \subset \mathcal{N}$,

(B) $\mathcal{N}$ is parallel, i.e. $\nabla_X \mathcal{N} \subset \mathcal{N}$ for all $X \in \mathrm{T}M^n$.

If g satisfies condition (A) it is called *null Ricci*. If g satisfies condition (B) it is called *Walker*. Metrics g satisfying both conditions (A) and (B) are called *null Ricci Walker*.

Fact

Every Walker metric (hence every null Ricci Walker metric) with its defining totally null plane $\mathcal{N}$ has *integrable* $\mathcal{N}^\perp$.

QUESTION: What if I restrict to null Ricci Walker metrics and take h such that its image is in the same $\mathcal{N}$ as image of $\mathbf{P}$?

IMMEDIATE ANSWER: Fefferman-Graham equations will be at most *quadratic* in h. But...maybe more...

Null Ricci Walker metrics

Let $\mathbf{P}$ be a Schouten tensor for a metric g on M^n and let $\mathcal{N}$ be a totally null distribution on M^n. Consider two conditions

(A) $Im(\mathbf{P}) \subset \mathcal{N}$,

(B) $\mathcal{N}$ is parallel, i.e. $\nabla_X \mathcal{N} \subset \mathcal{N}$ for all $X \in \mathrm{T}M^n$.

If g satisfies condition (A) it is called *null Ricci*. If g satisfies condition (B) it is called *Walker*. Metrics g satisfying both conditions (A) and (B) are called *null Ricci Walker*.

Fact

Every Walker metric (hence every null Ricci Walker metric) with its defining totally null plane $\mathcal{N}$ has *integrable* $\mathcal{N}^\perp$.

QUESTION: What if I restrict to null Ricci Walker metrics and take h such that its image is in the same $\mathcal{N}$ as image of $\mathbf{P}$?

IMMEDIATE ANSWER: Fefferman-Graham equations will be at most *quadratic* in h. But...maybe more...

Null Ricci Walker metrics

Let $\mathbf{P}$ be a Schouten tensor for a metric g on M^n and let $\mathcal{N}$ be a totally null distribution on M^n. Consider two conditions

(A) $Im(\mathbf{P}) \subset \mathcal{N}$,

(B) $\mathcal{N}$ is parallel, i.e. $\nabla_X \mathcal{N} \subset \mathcal{N}$ for all $X \in \mathrm{T}M^n$.

If g satisfies condition (A) it is called *null Ricci*. If g satisfies condition (B) it is called *Walker*. Metrics g satisfying both conditions (A) and (B) are called *null Ricci Walker*.

Fact

Every Walker metric (hence every null Ricci Walker metric) with its defining totally null plane $\mathcal{N}$ has *integrable* $\mathcal{N}^\perp$.

QUESTION: What if I restrict to null Ricci Walker metrics and take h such that its image is in the same $\mathcal{N}$ as image of $\mathbf{P}$?

IMMEDIATE ANSWER: Fefferman-Graham equations will be at most *quadratic* in h. But...maybe more...

Null Ricci Walker metrics

Let $\mathbf{P}$ be a Schouten tensor for a metric g on M^n and let $\mathcal{N}$ be a totally null distribution on M^n. Consider two conditions

(A) $Im(\mathbf{P}) \subset \mathcal{N}$,

(B) $\mathcal{N}$ is parallel, i.e. $\nabla_X \mathcal{N} \subset \mathcal{N}$ for all $X \in \mathrm{T}M^n$.

If g satisfies condition (A) it is called *null Ricci*. If g satisfies condition (B) it is called *Walker*. Metrics g satisfying both conditions (A) and (B) are called *null Ricci Walker*.

Fact

Every Walker metric (hence every null Ricci Walker metric) with its defining totally null plane $\mathcal{N}$ has *integrable* $\mathcal{N}^\perp$.

QUESTION: What if I restrict to null Ricci Walker metrics and take h such that its image is in the same $\mathcal{N}$ as image of $\mathbf{P}$?

IMMEDIATE ANSWER: Fefferman-Graham equations will be at most *quadratic* in h. But...maybe more...

Null Ricci Walker metrics and conformal holonomy

Theorem

A conformal holonomy of $[g]$ on a manifold M^n admits an invariant totally null subspace V of dimension $k+1$, with $k>0$, if and only if the conformal class $[g]$ contains a null Ricci Walker metric g with $\mathcal{N}$ of rank k.

Null Ricci Walker metrics and conformal holonomy

Theorem

A conformal holonomy of $[g]$ on a manifold M^n admits an invariant totally null subspace V of dimension $k+1$, with $k>0$, if and only if the conformal class $[g]$ contains a null Ricci Walker metric g with $\mathcal{N}$ of rank k.

Null Ricci Walker metrics and divergencefree h

Theorem

Let (M, g) be a null Ricci Walker manifold with parallel totally null distribution $\mathcal{N}$ such that $Im(\mathbf{P}) \subset \mathcal{N}$. Then the metric $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ with *divergencefree* h such that $Im(h) \subset \mathcal{N}$, satisfies $Ric(\tilde{g}) = 0$ if and only if h satisfies the following system of PDEs:

$$Lh_{ij} - \nabla^k \nabla_k h_{ij} + 2R^k{}_{ij}{}^l h_{kl} + 2R_{ij} + h^{kl}\nabla_k \nabla_l h_{ij} + \nabla_k h_{li} \nabla^l h^k{}_j = 0,$$

with $L = 2\rho \frac{\partial^2}{\partial \rho^2} - (n-2)\frac{\partial}{\partial \rho}$.

Null Ricci Walker metrics and divergencefree h

Theorem

Let (M, g) be a null Ricci Walker manifold with parallel totally null distribution $\mathcal{N}$ such that $Im(\mathbf{P}) \subset \mathcal{N}$. Then the metric $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ with *divergencefree* h such that $Im(h) \subset \mathcal{N}$, satisfies $Ric(\tilde{g}) = 0$ if and only if h satisfies the following system of PDEs:

$$Lh_{ij} - \nabla^k \nabla_k h_{ij} + 2R^k{}_{ij}{}^l h_{kl} + 2R_{ij} + h^{kl} \nabla_k \nabla_l h_{ij} + \nabla_k h_{li} \nabla^l h^k{}_j = 0,$$

with $L = 2\rho \frac{\partial^2}{\partial \rho^2} - (n-2)\frac{\partial}{\partial \rho}$.

Null Ricci Walker metrics and divergencefree h

Theorem
Let (M, g) be a null Ricci Walker manifold with parallel totally null distribution $\mathcal{N}$ such that $Im(\mathbf{P}) \subset \mathcal{N}$. Then the metric $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ with *divergencefree* h such that $Im(h) \subset \mathcal{N}$, satisfies $Ric(\tilde{g}) = 0$ if and only if h satisfies the following system of PDEs:

$$Lh_{ij} - \nabla^k \nabla_k h_{ij} + 2R^k{}_{ij}{}^l h_{kl} + 2R_{ij} + h^{kl}\nabla_k\nabla_l h_{ij} + \nabla_k h_{li}\nabla^l h^k{}_j = 0,$$

with $L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho}$.

Null Ricci Walker metrics with $\mathcal{N}$ of rank 1

Theorem

Let (M, g) be a null Ricci Walker manifold with parallel totally null *line* $\mathcal{N}$ such that $Im(\mathbf{P}) \subset \mathcal{N}$. Then the metric $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ with h such that $Im(h) \subset \mathcal{N}$ and such that $\mathcal{L}_X h = 0$ for all X alligned with $\mathcal{N}$, satisfies $Ric(\tilde{g}) = 0$ if and only if h satisfies the following system of LINEAR PDEs:

$$Lh_{ij} - \nabla^k \nabla_k h_{ij} + 2R^k{}_{ij}{}^l h_{kl} + 2R_{ij} = 0,$$

with $L = 2\rho \frac{\partial^2}{\partial \rho^2} - (n-2)\frac{\partial}{\partial \rho}$.

Null Ricci Walker metrics with $\mathcal{N}$ of rank 1

Theorem
Let (M, g) be a null Ricci Walker manifold with parallel totally null *line* $\mathcal{N}$ such that $Im(\mathbf{P}) \subset \mathcal{N}$. Then the metric $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ with h such that $Im(h) \subset \mathcal{N}$ and such that $\mathcal{L}_X h = 0$ for all X alligned with $\mathcal{N}$, satisfies $Ric(\tilde{g}) = 0$ if and only if h satisfies the following system of LINEAR PDEs:

$$Lh_{ij} - \nabla^k \nabla_k h_{ij} + 2R^k{}_{ij}{}^l h_{kl} + 2R_{ij} = 0,$$

with $L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho}$.

Null Ricci Walker metrics with $\mathcal{N}$ of rank 1

Theorem

Let (M, g) be a null Ricci Walker manifold with parallel totally null *line* $\mathcal{N}$ such that $Im(\mathbf{P}) \subset \mathcal{N}$. Then the metric $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ with h such that $Im(h) \subset \mathcal{N}$ and such that $\mathcal{L}_X h = 0$ for all X alligned with $\mathcal{N}$, satisfies $Ric(\tilde{g}) = 0$ if and only if h satisfies the following system of LINEAR PDEs:

$$Lh_{ij} - \nabla^k \nabla_k h_{ij} + 2R^k{}_{ij}{}^l h_{kl} + 2R_{ij} = 0,$$

with $L = 2\rho \frac{\partial^2}{\partial \rho^2} - (n-2)\frac{\partial}{\partial \rho}$.

Null Ricci Walker metrics with $Riemann(\mathcal{N}, ., ., .) = 0$

Theorem

Let (M^n, g) be a null Ricci Walker manifold with parallel totally null *plane* $\mathcal{N}$ such that $Im(\mathbf{P}) \subset \mathcal{N}$ and such that $Riemann(\mathcal{N}, ., ., .) = 0$. Then if the metric $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ is an ambient metric for g then $\mathcal{L}_X h = O(\rho^m)$ for all $X \in \mathcal{N}$. Here $m = \infty$ if n is odd, and $n = 2m$ otherwise. In such a case the Fefferman-Graham equations for h are again given by linear PDEs:

$$Lh_{ij} - \nabla^k \nabla_k h_{ij} + 2R^k{}_{ij}{}^l h_{kl} + 2R_{ij} = 0,$$

with $L = 2\rho \frac{\partial^2}{\partial \rho^2} - (n-2)\frac{\partial}{\partial \rho}$.

Null Ricci Walker metrics with $Riemann(\mathcal{N},.,.,.) = 0$

Theorem

Let (M^n, g) be a null Ricci Walker manifold with parallel totally null *plane* $\mathcal{N}$ such that $Im(\mathbf{P}) \subset \mathcal{N}$ and such that $Riemann(\mathcal{N},.,.,.) = 0$. Then if the metric $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ is an ambient metric for g then $\mathcal{L}_X h = O(\rho^m)$ for all $X \in \mathcal{N}$. Here $m = \infty$ if n is odd, and $n = 2m$ otherwise. In such a case the Fefferman-Graham equations for h are again given by linear PDEs:

$$Lh_{ij} - \nabla^k \nabla_k h_{ij} + 2R^k{}_{ij}{}^l h_{kl} + 2R_{ij} = 0,$$

with $L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho}$.

Null Ricci Walker metrics with $Riemann(\mathcal{N},.,.,.)=0$

Theorem

Let (M^n, g) be a null Ricci Walker manifold with parallel totally null *plane* $\mathcal{N}$ such that $Im(\mathbf{P}) \subset \mathcal{N}$ and such that $Riemann(\mathcal{N},.,.,.)=0$. Then if the metric $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g+h)$ is an ambient metric for g then $\mathcal{L}_X h = O(\rho^m)$ for all $X \in \mathcal{N}$. Here $m=\infty$ if n is odd, and $n=2m$ otherwise. In such a case the Fefferman-Graham equations for h are again given by linear PDEs:

$$Lh_{ij} - \nabla^k \nabla_k h_{ij} + 2R^k{}_{ij}{}^l h_{kl} + 2R_{ij} = 0,$$

with $L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho}$.

Null Ricci Walker metrics with $Riemann(\mathcal{N},.,.,.)=0$

Theorem

Let (M^n, g) be a null Ricci Walker manifold with parallel totally null *plane* $\mathcal{N}$ such that $Im(\mathbf{P}) \subset \mathcal{N}$ and such that $Riemann(\mathcal{N},.,.,.)=0$. Then if the metric $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g+h)$ is an ambient metric for g then $\mathcal{L}_X h = O(\rho^m)$ for all $X \in \mathcal{N}$. Here $m=\infty$ if n is odd, and $n=2m$ otherwise. In such a case the Fefferman-Graham equations for h are again given by linear PDEs:

$$Lh_{ij} - \nabla^k\nabla_k h_{ij} + 2R^k{}_{ij}{}^l h_{kl} + 2R_{ij} = 0,$$

with $L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho}$.

Null Ricci Walker metrics with $Riemann(\mathcal{N},.,.,.) = 0$

Theorem

Let (M^n, g) be a null Ricci Walker manifold with parallel totally null *plane* $\mathcal{N}$ such that $Im(\mathbf{P}) \subset \mathcal{N}$ and such that $Riemann(\mathcal{N},.,.,.) = 0$. Then if the metric $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ is an ambient metric for g then $\mathcal{L}_X h = O(\rho^m)$ for all $X \in \mathcal{N}$. Here $m = \infty$ if n is odd, and $n = 2m$ otherwise. In such a case the Fefferman-Graham equations for h are again given by linear PDEs:

$$Lh_{ij} - \nabla^k \nabla_k h_{ij} + 2R^k{}_{ij}{}^l h_{kl} + 2R_{ij} = 0,$$

with $L = 2\rho \frac{\partial^2}{\partial \rho^2} - (n-2)\frac{\partial}{\partial \rho}$.

In which class are G_2 examples?

- It turns out that all known examples of not conformally Einstein metrics with explicit Fefferman-Graham metrics correspond to null Ricci Walker metrics, EXCEPT the examples associated to $(2,3,5)$ distributions.
- The metrics g of these examples are null Ricci (satisfy condition (A), i.e. $Im(\mathbf{P} \subset \mathcal{N})$, but they are NOT Walker (condition (B) is NOT satisfied, i.e. $\mathcal{N}$ is not parallel).
- It is convenient to give a label, say (C), to a condition which says that in the metric g the distribution $\mathcal{N}^\perp$ orthogonal to the totally null distribution N is *integrable*, $[\mathcal{N}^\perp, \mathcal{N}^\perp] \subset \mathcal{N}^\perp$.
- Metrics g which satisfy both (A) and (C), are null Ricci metrics with integrable $\mathcal{N}^\perp$ and $Im(\mathbf{P}) \subset \mathcal{N}$.

In which class are G_2 examples?

- It turns out that all known examples of not conformally Einstein metrics with explicit Fefferman-Graham metrics correspond to null Ricci Walker metrics, EXCEPT the examples associated to $(2,3,5)$ distributions.
- The metrics g of these examples are null Ricci (satisfy condition (A), i.e. $Im(\mathbf{P} \subset \mathcal{N})$, but they are NOT Walker (condition (B) is NOT satisfied, i.e. $\mathcal{N}$ is not parallel).
- It is convenient to give a label, say (C), to a condition which says that in the metric g the distribution $\mathcal{N}^\perp$ orthogonal to the totally null distribution N is *integrable*, $[\mathcal{N}^\perp, \mathcal{N}^\perp] \subset \mathcal{N}^\perp$.
- Metrics g which satisfy both (A) and (C), are null Ricci metrics with integrable $\mathcal{N}^\perp$ and $Im(\mathbf{P}) \subset \mathcal{N}$.

In which class are G_2 examples?

- It turns out that all known examples of not conformally Einstein metrics with explicit Fefferman-Graham metrics correspond to null Ricci Walker metrics, EXCEPT the examples associated to $(2, 3, 5)$ distributions.
- The metrics g of these examples are null Ricci (satisfy condition (A), i.e. $Im(\mathbf{P} \subset \mathcal{N})$, but they are NOT Walker (condition (B) is NOT satisfied, i.e. $\mathcal{N}$ is not parallel).
- It is convenient to give a label, say (C), to a condition which says that in the metric g the distribution $\mathcal{N}^\perp$ orthogonal to the totally null distribution N is *integrable*, $[\mathcal{N}^\perp, \mathcal{N}^\perp] \subset \mathcal{N}^\perp$.
- Metrics g which satisfy both (A) and (C), are null Ricci metrics with integrable $\mathcal{N}^\perp$ and $Im(\mathbf{P}) \subset \mathcal{N}$.

In which class are G_2 examples?

- It turns out that all known examples of not conformally Einstein metrics with explicit Fefferman-Graham metrics correspond to null Ricci Walker metrics, EXCEPT the examples associated to $(2, 3, 5)$ distributions.
- The metrics g of these examples are null Ricci (satisfy condition (A), i.e. $Im(\mathbf{P} \subset \mathcal{N})$, but they are NOT Walker (condition (B) is NOT satisfied, i.e. $\mathcal{N}$ is not parallel).
- It is convenient to give a label, say (C), to a condition which says that in the metric g the distribution $\mathcal{N}^\perp$ orthogonal to the totally null distribution N is *integrable*, $[\mathcal{N}^\perp, \mathcal{N}^\perp] \subset \mathcal{N}^\perp$.
- Metrics g which satisfy both (A) and (C), are null Ricci metrics with integrable $\mathcal{N}^\perp$ and $Im(\mathbf{P}) \subset \mathcal{N}$.

In which class are G_2 examples?

- It turns out that all known examples of not conformally Einstein metrics with explicit Fefferman-Graham metrics correspond to null Ricci Walker metrics, EXCEPT the examples associated to $(2,3,5)$ distributions.
- The metrics g of these examples are null Ricci (satisfy condition (A), i.e. $Im(\mathbf{P} \subset \mathcal{N})$, but they are NOT Walker (condition (B) is NOT satisfied, i.e. $\mathcal{N}$ is not parallel).
- It is convenient to give a label, say (C), to a condition which says that in the metric g the distribution $\mathcal{N}^\perp$ orthogonal to the totally null distribution N is *integrable*, $[\mathcal{N}^\perp, \mathcal{N}^\perp] \subset \mathcal{N}^\perp$.
- Metrics g which satisfy both (A) and (C), are null Ricci metrics with integrable $\mathcal{N}^\perp$ and $Im(\mathbf{P}) \subset \mathcal{N}$.

In which class are G_2 examples?

Theorem

Let (M, g) be a null Ricci manifold with $Im(\mathbf{P}) \subset \mathcal{N}$ and integrable $\mathcal{N}^{\perp}$. If in addition $L_X\mathbf{P} = 0$ for each $X \in \mathcal{N}$, then the metric $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ with h such that $Im(h) \subset \mathcal{N}$ and such that $\mathcal{L}_X h = 0$ for all $X \in \mathcal{N}$, satisfies $Ric(\tilde{g}) = 0$ if and only if h satisfies the following system of LINEAR PDEs:

$$Lh_{ij} - \nabla^k\nabla_k h_{ij} + 2R^k{}_{ij}{}^l h_{kl} + 2R_{ij} = 0,$$

with $L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho}$.

It is this class of g in which seats our $g_{\mathcal{D}}$ for a $(2, 3, 5)$ distribution $\mathcal{D}$. This theorem describes why the FG equations are linear in this case.

In which class are G_2 examples?

Theorem

Let (M, g) be a null Ricci manifold with $Im(\mathbf{P}) \subset \mathcal{N}$ and integrable $\mathcal{N}^{\perp}$. If in addition $L_X\mathbf{P} = 0$ for each $X \in \mathcal{N}$, then the metric $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ with h such that $Im(h) \subset \mathcal{N}$ and such that $\mathcal{L}_X h = 0$ for all $X \in \mathcal{N}$, satisfies $Ric(\tilde{g}) = 0$ if and only if h satisfies the following system of LINEAR PDEs:

$$Lh_{ij} - \nabla^k\nabla_k h_{ij} + 2R^k{}_{ij}{}^l h_{kl} + 2R_{ij} = 0,$$

with $L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho}$.

It is this class of g in which seats our $g_{\mathcal{D}}$ for a $(2, 3, 5)$ distribution $\mathcal{D}$. This theorem describes why the FG equations are linear in this case.

In which class are G_2 examples?

Theorem

Let (M, g) be a null Ricci manifold with $Im(\mathbf{P}) \subset \mathcal{N}$ and integrable $\mathcal{N}^{\perp}$. If in addition $L_X \mathbf{P} = 0$ for each $X \in \mathcal{N}$, then the metric $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ with h such that $Im(h) \subset \mathcal{N}$ and such that $\mathcal{L}_X h = 0$ for all $X \in \mathcal{N}$, satisfies $Ric(\tilde{g}) = 0$ if and only if h satisfies the following system of LINEAR PDEs:

$$Lh_{ij} - \nabla^k \nabla_k h_{ij} + 2R^k{}_{ij}{}^l h_{kl} + 2R_{ij} = 0,$$

with $L = 2\rho \frac{\partial^2}{\partial \rho^2} - (n-2)\frac{\partial}{\partial \rho}$.

It is this class of g in which seats our $g_{\mathcal{D}}$ for a $(2, 3, 5)$ distribution $\mathcal{D}$. This theorem describes why the FG equations are linear in this case.

In which class are G_2 examples?

Theorem

Let (M, g) be a null Ricci manifold with $Im(\mathbf{P}) \subset \mathcal{N}$ and integrable $\mathcal{N}^{\perp}$. If in addition $L_X\mathbf{P} = 0$ for each $X \in \mathcal{N}$, then the metric $\tilde{g} = \mathrm{d}t\mathrm{d}(\rho t) + t^2(g + h)$ with h such that $Im(h) \subset \mathcal{N}$ and such that $\mathcal{L}_X h = 0$ for all $X \in \mathcal{N}$, satisfies $Ric(\tilde{g}) = 0$ if and only if h satisfies the following system of LINEAR PDEs:

$$Lh_{ij} - \nabla^k\nabla_k h_{ij} + 2R^k{}_{ij}{}^l h_{kl} + 2R_{ij} = 0,$$

with $L = 2\rho\frac{\partial^2}{\partial\rho^2} - (n-2)\frac{\partial}{\partial\rho}$.

It is this class of g in which seats our $g_{\mathcal{D}}$ for a $(2, 3, 5)$ distribution $\mathcal{D}$. This theorem describes why the FG equations are linear in this case.

Plan

Explicit FG metrics for conformal classes on groups

Interpreting conditions from our 'the best theorem we have' in terms of commutators of bases of $\mathcal{N}$, $\mathcal{N}^{\perp}$ and its complement to $\mathrm{T}M$, we construct certain left invariant metrics on certain groups. The structure of these groups were deduced from these commutators.

- Take $\mathfrak{n}$ to be a 2-step nilpotent Lie algebra. This means that $\mathfrak{n} = \mathfrak{m} \oplus \mathfrak{z}$ with $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{z}$, and $\mathfrak{z}$ is the center of $\mathfrak{n}$. We set $\dim \mathfrak{z} = p < q = \dim \mathfrak{z}$.
- Take a Lie group H of $\dim H = p$, and let $\mathfrak{h}$ be the Lie algebra of H.
- Take ANY homomorphism $\phi : \mathfrak{h} \to \mathfrak{der}(\mathfrak{n})$.
- Define $\mathfrak{g}$ as semidirect product of $\mathfrak{h}$ with $\mathfrak{n}$ by the homomorphism ϕ, $\mathfrak{g} = \mathfrak{h} \ltimes_{\phi} \mathfrak{n} = \mathfrak{h} \ltimes_{\phi} (\mathfrak{m} \oplus \mathfrak{z})$.

Explicit FG metrics for conformal classes on groups

Interpreting conditions from our 'the best theorem we have' in terms of commutators of bases of $\mathcal{N}$, $\mathcal{N}^\perp$ and its complement to $\mathrm{T}M$, we construct certain left invariant metrics on certain groups. The structure of these groups were deduced from these commutators.

- Take $\mathfrak{n}$ to be a 2-step nilpotent Lie algebra. This means that $\mathfrak{n} = \mathfrak{m} \oplus \mathfrak{z}$ with $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{z}$, and $\mathfrak{z}$ is the center of $\mathfrak{n}$. We set $\dim \mathfrak{z} = p < q = \dim \mathfrak{z}$.
- Take a Lie group H of $\dim H = p$, and let $\mathfrak{h}$ be the Lie algebra of H.
- Take ANY homomorphism $\phi : \mathfrak{h} \to \mathfrak{der}(\mathfrak{n})$.
- Define $\mathfrak{g}$ as semidirect product of $\mathfrak{h}$ with $\mathfrak{n}$ by the homomorphism ϕ, $\mathfrak{g} = \mathfrak{h} \ltimes_\phi \mathfrak{n} = \mathfrak{h} \ltimes_\phi (\mathfrak{m} \oplus \mathfrak{z})$.

Explicit FG metrics for conformal classes on groups

Interpreting conditions from our 'the best theorem we have' in terms of commutators of bases of $\mathcal{N}$, $\mathcal{N}^{\perp}$ and its complement to $\mathrm{T}M$, we construct certain left invariant metrics on certain groups. The structure of these groups were deduced from these commutators.

- Take $\mathfrak{n}$ to be a 2-step nilpotent Lie algebra. This means that $\mathfrak{n} = \mathfrak{m} \oplus \mathfrak{z}$ with $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{z}$, and $\mathfrak{z}$ is the center of $\mathfrak{n}$. We set $\dim \mathfrak{z} = p < q = \dim \mathfrak{z}$.
- Take a Lie group H of $\dim H = p$, and let $\mathfrak{h}$ be the Lie algebra of H.
- Take ANY homomorphism $\phi : \mathfrak{h} \to \mathfrak{der}(\mathfrak{n})$.
- Define $\mathfrak{g}$ as semidirect product of $\mathfrak{h}$ with $\mathfrak{n}$ by the homomorphism ϕ, $\mathfrak{g} = \mathfrak{h} \ltimes_{\phi} \mathfrak{n} = \mathfrak{h} \ltimes_{\phi} (\mathfrak{m} \oplus \mathfrak{z})$.

Explicit FG metrics for conformal classes on groups

Interpreting conditions from our 'the best theorem we have' in terms of commutators of bases of $\mathcal{N}$, $\mathcal{N}^{\perp}$ and its complement to $\mathrm{T}M$, we construct certain left invariant metrics on certain groups. The structure of these groups were deduced from these commutators.

- Take $\mathfrak{n}$ to be a 2-step nilpotent Lie algebra. This means that $\mathfrak{n} = \mathfrak{m} \oplus \mathfrak{z}$ with $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{z}$, and $\mathfrak{z}$ is the center of $\mathfrak{n}$. We set $\dim \mathfrak{z} = p < q = \dim \mathfrak{z}$.
- Take a Lie group H of $\dim H = p$, and let $\mathfrak{h}$ be the Lie algebra of H.
- Take ANY homomorphism $\phi : \mathfrak{h} \to \mathfrak{der}(\mathfrak{n})$.
- Define $\mathfrak{g}$ as semidirect product of $\mathfrak{h}$ with $\mathfrak{n}$ by the homomorphism ϕ, $\mathfrak{g} = \mathfrak{h} \ltimes_{\phi} \mathfrak{n} = \mathfrak{h} \ltimes_{\phi} (\mathfrak{m} \oplus \mathfrak{z})$.

Explicit FG metrics for conformal classes on groups

Interpreting conditions from our 'the best theorem we have' in terms of commutators of bases of $\mathcal{N}$, $\mathcal{N}^\perp$ and its complement to $\mathrm{T}M$, we construct certain left invariant metrics on certain groups. The structure of these groups were deduced from these commutators.

- Take $\mathfrak{n}$ to be a 2-step nilpotent Lie algebra. This means that $\mathfrak{n} = \mathfrak{m} \oplus \mathfrak{z}$ with $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{z}$, and $\mathfrak{z}$ is the center of $\mathfrak{n}$. We set $\dim \mathfrak{z} = p < q = \dim \mathfrak{z}$.
- Take a Lie group H of $\dim H = p$, and let $\mathfrak{h}$ be the Lie algebra of H.
- Take ANY homomorphism $\phi : \mathfrak{h} \to \mathfrak{der}(\mathfrak{n})$.
- Define $\mathfrak{g}$ as semidirect product of $\mathfrak{h}$ with $\mathfrak{n}$ by the homomorphism ϕ, $\mathfrak{g} = \mathfrak{h} \ltimes_\phi \mathfrak{n} = \mathfrak{h} \ltimes_\phi (\mathfrak{m} \oplus \mathfrak{z})$.

Explicit FG metrics for conformal classes on groups

Interpreting conditions from our 'the best theorem we have' in terms of commutators of bases of $\mathcal{N}$, $\mathcal{N}^{\perp}$ and its complement to $\mathrm{T}M$, we construct certain left invariant metrics on certain groups. The structure of these groups were deduced from these commutators.

- Take $\mathfrak{n}$ to be a 2-step nilpotent Lie algebra. This means that $\mathfrak{n} = \mathfrak{m} \oplus \mathfrak{z}$ with $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{z}$, and $\mathfrak{z}$ is the center of $\mathfrak{n}$. We set $\dim \mathfrak{z} = p < q = \dim \mathfrak{z}$.
- Take a Lie group H of $\dim H = p$, and let $\mathfrak{h}$ be the Lie algebra of H.
- Take ANY homomorphism $\phi : \mathfrak{h} \to \mathfrak{der}(\mathfrak{n})$.
- Define $\mathfrak{g}$ as semidirect product of $\mathfrak{h}$ with $\mathfrak{n}$ by the homomorphism ϕ, $\mathfrak{g} = \mathfrak{h} \ltimes_{\phi} \mathfrak{n} = \mathfrak{h} \ltimes_{\phi} (\mathfrak{m} \oplus \mathfrak{z})$.

Explicit FG metrics for conformal classes on groups

- Take a Lie group G with Lie algebra $\mathfrak{g} = \mathfrak{h} \ltimes_\phi (\mathfrak{z} \oplus \mathfrak{m})$, and a basis $(\mathbf{e}_\mu)$ of left invariant vector fields on G.
- We have $(\mathbf{e}_\mu) = (\mathbf{e}_{\bar{a}}, \mathbf{e}_A, \mathbf{e}_a)$ with
  - $(\mathbf{e}_{\bar{a}})$, $\bar{a} = 1, 2, \ldots, p$ basis for $\mathfrak{h}$,
  - $(\mathbf{e}_a)$, $\bar{a} = 1, 2, \ldots, p$ basis for $\mathfrak{z}$,
  - $(\mathbf{e}_A)$, $A = p+1, p+2, \ldots, q$ basis for $\mathfrak{m}$.
- Let $\theta^\mu = (\theta^{\bar{a}}, \theta^A, \theta^a)$ be the dual basis to $(\mathbf{e}_\mu) = (\mathbf{e}_{\bar{a}}, \mathbf{e}_A, \mathbf{e}_a)$.
- Define $g = 2g_{a\bar{c}}\theta^a\theta^{\bar{c}} + 2g_{AB}\theta^A\theta^B$ with $g_{a\bar{c}}$ and g_{AB} real constants so that the symmetric bilinear form g on G is nondegenerate.

Explicit FG metrics for conformal classes on groups

- Take a Lie group G with Lie algebra $\mathfrak{g} = \mathfrak{h} \ltimes_\phi (\mathfrak{z} \oplus \mathfrak{m})$, and a basis $(\mathbf{e}_\mu)$ of left invariant vector fields on G.
- We have $(\mathbf{e}_\mu) = (\mathbf{e}_{\bar{a}}, \mathbf{e}_A, \mathbf{e}_a)$ with
  - $(\mathbf{e}_{\bar{a}})$, $\bar{a} = 1, 2, \ldots, p$ basis for $\mathfrak{h}$,
  - $(\mathbf{e}_a)$, $\bar{a} = 1, 2, \ldots, p$ basis for $\mathfrak{z}$,
  - $(\mathbf{e}_A)$, $A = p+1, p+2, \ldots, q$ basis for $\mathfrak{m}$.
- Let $\theta^\mu = (\theta^{\bar{a}}, \theta^A, \theta^a)$ be the dual basis to $(\mathbf{e}_\mu) = (\mathbf{e}_{\bar{a}}, \mathbf{e}_A, \mathbf{e}_a)$.
- Define $g = 2g_{a\bar{c}}\theta^a\theta^{\bar{c}} + 2g_{AB}\theta^A\theta^B$ with $g_{a\bar{c}}$ and g_{AB} real constants so that the symmetric bilinear form g on G is nondegenerate.

Explicit FG metrics for conformal classes on groups

- Take a Lie group G with Lie algebra $\mathfrak{g} = \mathfrak{h} \ltimes_\phi (\mathfrak{z} \oplus \mathfrak{m})$, and a basis $(\mathbf{e}_\mu)$ of left invariant vector fields on G.
- We have $(\mathbf{e}_\mu) = (\mathbf{e}_{\bar{a}}, \mathbf{e}_A, \mathbf{e}_a)$ with
  - $(\mathbf{e}_{\bar{a}})$, $\bar{a} = 1, 2, \ldots, p$ basis for $\mathfrak{h}$,
  - $(\mathbf{e}_a)$, $\bar{a} = 1, 2, \ldots, p$ basis for $\mathfrak{z}$,
  - $(\mathbf{e}_A)$, $A = p+1, p+2, \ldots, q$ basis for $\mathfrak{m}$.
- Let $\theta^\mu = (\theta^{\bar{a}}, \theta^A, \theta^a)$ be the dual basis to $(\mathbf{e}_\mu) = (\mathbf{e}_{\bar{a}}, \mathbf{e}_A, \mathbf{e}_a)$.
- Define $g = 2g_{a\bar{c}}\theta^a\theta^{\bar{c}} + 2g_{AB}\theta^A\theta^B$ with $g_{a\bar{c}}$ and g_{AB} real constants so that the symmetric bilinear form g on G is nondegenerate.

Explicit FG metrics for conformal classes on groups

- Take a Lie group G with Lie algebra $\mathfrak{g} = \mathfrak{h} \ltimes_\phi (\mathfrak{z} \oplus \mathfrak{m})$, and a basis $(\mathbf{e}_\mu)$ of left invariant vector fields on G.
- We have $(\mathbf{e}_\mu) = (\mathbf{e}_{\bar{a}}, \mathbf{e}_A, \mathbf{e}_a)$ with
  - $(\mathbf{e}_{\bar{a}})$, $\bar{a} = 1, 2, \ldots, p$ basis for $\mathfrak{h}$,
  - $(\mathbf{e}_a)$, $\bar{a} = 1, 2, \ldots, p$ basis for $\mathfrak{z}$,
  - $(\mathbf{e}_A)$, $A = p+1, p+2, \ldots, q$ basis for $\mathfrak{m}$.
- Let $\theta^\mu = (\theta^{\bar{a}}, \theta^A, \theta^a)$ be the dual basis to $(\mathbf{e}_\mu) = (\mathbf{e}_{\bar{a}}, \mathbf{e}_A, \mathbf{e}_a)$.
- Define $g = 2g_{a\bar{c}}\theta^a\theta^{\bar{c}} + 2g_{AB}\theta^A\theta^B$ with $g_{a\bar{c}}$ and g_{AB} real constants so that the symmetric bilinear form g on G is nondegenerate.

Explicit FG metrics for conformal classes on groups

- Take a Lie group G with Lie algebra $\mathfrak{g} = \mathfrak{h} \ltimes_\phi (\mathfrak{z} \oplus \mathfrak{m})$, and a basis $(\mathbf{e}_\mu)$ of left invariant vector fields on G.
- We have $(\mathbf{e}_\mu) = (\mathbf{e}_{\bar{a}}, \mathbf{e}_A, \mathbf{e}_a)$ with
    - $(\mathbf{e}_{\bar{a}})$, $\bar{a} = 1, 2, \ldots, p$ basis for $\mathfrak{h}$,
    - $(\mathbf{e}_a)$, $\bar{a} = 1, 2, \ldots, p$ basis for $\mathfrak{z}$,
    - $(\mathbf{e}_A)$, $A = p+1, p+2, \ldots, q$ basis for $\mathfrak{m}$.
- Let $\theta^\mu = (\theta^{\bar{a}}, \theta^A, \theta^a)$ be the dual basis to $(\mathbf{e}_\mu) = (\mathbf{e}_{\bar{a}}, \mathbf{e}_A, \mathbf{e}_a)$.
- Define $g = 2g_{a\bar{c}}\theta^a\theta^{\bar{c}} + 2g_{AB}\theta^A\theta^B$ with $g_{a\bar{c}}$ and g_{AB} real constants so that the symmetric bilinear form g on G is nondegenerate.

Explicit FG metrics for conformal classes on groups

- Take a Lie group G with Lie algebra $\mathfrak{g} = \mathfrak{h} \ltimes_\phi (\mathfrak{z} \oplus \mathfrak{m})$, and a basis $(\mathbf{e}_\mu)$ of left invariant vector fields on G.
- We have $(\mathbf{e}_\mu) = (\mathbf{e}_{\bar{a}}, \mathbf{e}_A, \mathbf{e}_a)$ with
  - $(\mathbf{e}_{\bar{a}})$, $\bar{a} = 1, 2, \ldots, p$ basis for $\mathfrak{h}$,
  - $(\mathbf{e}_a)$, $\bar{a} = 1, 2, \ldots, p$ basis for $\mathfrak{z}$,
  - $(\mathbf{e}_A)$, $A = p + 1, p + 2, \ldots, q$ basis for $\mathfrak{m}$.
- Let $\theta^\mu = (\theta^{\bar{a}}, \theta^A, \theta^a)$ be the dual basis to $(\mathbf{e}_\mu) = (\mathbf{e}_{\bar{a}}, \mathbf{e}_A, \mathbf{e}_a)$.
- Define $g = 2g_{a\bar{c}}\theta^a\theta^{\bar{c}} + 2g_{AB}\theta^A\theta^B$ with $g_{a\bar{c}}$ and g_{AB} real constants so that the symmetric bilinear form g on G is nondegenerate.

Explicit FG metrics for conformal classes on groups

- Take a Lie group G with Lie algebra $\mathfrak{g} = \mathfrak{h} \ltimes_\phi (\mathfrak{z} \oplus \mathfrak{m})$, and a basis $(\mathbf{e}_\mu)$ of left invariant vector fields on G.
- We have $(\mathbf{e}_\mu) = (\mathbf{e}_{\bar{a}}, \mathbf{e}_A, \mathbf{e}_a)$ with
  - $(\mathbf{e}_{\bar{a}})$, $\bar{a} = 1, 2, \ldots, p$ basis for $\mathfrak{h}$,
  - $(\mathbf{e}_a)$, $\bar{a} = 1, 2, \ldots, p$ basis for $\mathfrak{z}$,
  - $(\mathbf{e}_A)$, $A = p+1, p+2, \ldots, q$ basis for $\mathfrak{m}$.
- Let $\theta^\mu = (\theta^{\bar{a}}, \theta^A, \theta^a)$ be the dual basis to $(\mathbf{e}_\mu) = (\mathbf{e}_{\bar{a}}, \mathbf{e}_A, \mathbf{e}_a)$.
- Define $g = 2g_{a\bar{c}}\theta^a\theta^{\bar{c}} + 2g_{AB}\theta^A\theta^B$ with $g_{a\bar{c}}$ and g_{AB} real constants so that the symmetric bilinear form g on G is nondegenerate.

Explicit FG metrics for conformal classes on groups

- Take a Lie group G with Lie algebra $\mathfrak{g} = \mathfrak{h} \ltimes_\phi (\mathfrak{z} \oplus \mathfrak{m})$, and a basis $(\mathbf{e}_\mu)$ of left invariant vector fields on G.
- We have $(\mathbf{e}_\mu) = (\mathbf{e}_{\bar{a}}, \mathbf{e}_A, \mathbf{e}_a)$ with
  - $(\mathbf{e}_{\bar{a}})$, $\bar{a} = 1, 2, \dots, p$ basis for $\mathfrak{h}$,
  - $(\mathbf{e}_a)$, $\bar{a} = 1, 2, \dots, p$ basis for $\mathfrak{z}$,
  - $(\mathbf{e}_A)$, $A = p+1, p+2, \dots, q$ basis for $\mathfrak{m}$.
- Let $\theta^\mu = (\theta^{\bar{a}}, \theta^A, \theta^a)$ be the dual basis to $(\mathbf{e}_\mu) = (\mathbf{e}_{\bar{a}}, \mathbf{e}_A, \mathbf{e}_a)$.
- Define $g = 2g_{a\bar{c}}\theta^a\theta^{\bar{c}} + 2g_{AB}\theta^A\theta^B$ with $g_{a\bar{c}}$ and g_{AB} real constants so that the symmetric bilinear form g on G is nondegenerate.

Explicit FG metrics for conformal classes on groups

Theorem

The metric g is a left invariant *null Ricci Walker* on G with parallel distribution $\mathcal{N}$ given by left invariant vector fields forming $\mathfrak{z}$. Moreover

$$\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2\left(g + \frac{2Ric(g)}{p+q-2}\rho\right)$$

satisfies $Ric(\tilde{g}) = 0$. In particular if $\dim(G) = p + q$ is even the Fefferman-Graham obstruction for $[g]$ vanishes.

Explicit FG metrics for conformal classes on groups

Theorem

The metric g is a left invariant *null Ricci Walker* on G with parallel distribution $\mathcal{N}$ given by left invariant vector fields forming $\mathfrak{z}$. Moreover

$$\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2\left(g + \frac{2Ric(g)}{p+q-2}\rho\right)$$

satisfies $Ric(\tilde{g}) = 0$. In particular if $\dim(G) = p + q$ is even the Fefferman-Graham obstruction for $[g]$ vanishes.

Explicit FG metrics for conformal classes on groups

Theorem

The metric g is a left invariant *null Ricci Walker* on G with parallel distribution $\mathcal{N}$ given by left invariant vector fields forming $\mathfrak{z}$. Moreover

$$\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2(g + \frac{2Ric(g)}{p+q-2}\rho)$$

satisfies $Ric(\tilde{g}) = 0$. In particular if $\dim(G) = p + q$ is even the Fefferman-Graham obstruction for $[g]$ vanishes.

Explicit FG metrics for conformal classes on groups

Theorem

The metric g is a left invariant *null Ricci Walker* on G with parallel distribution $\mathcal{N}$ given by left invariant vector fields forming $\mathfrak{z}$. Moreover

$$\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2\left(g + \frac{2Ric(g)}{p+q-2}\rho\right)$$

satisfies $Ric(\tilde{g}) = 0$. In particular if $\dim(G) = p+q$ is even the Fefferman-Graham obstruction for $[g]$ vanishes.

Happy Birthday Andreas!

THANK YOU!

Happy Birthday Andreas!

THANK YOU!