Problem Set 1

Differential Geometry
WS 2019/20

Problem 1
(a) Let $\tilde{\gamma} : [a, b] \to \mathbb{R}^n$ be a regular, differentiable curve and denote by $\gamma : [0, \ell(\gamma)] \to \mathbb{R}^n$ its arclength reparametrization. Show that γ is differentiable and $\|\dot{\gamma}(t)\| = 1$ for all $t \in [0, \ell(\gamma)]$.
(b) What can be said if $\tilde{\gamma}$ is not necessarily regular but nowhere constant?

Problem 2
(a) Let $\tilde{\gamma} : [a, b] \to \mathbb{R}^2$ be a regular, twice differentiable curve. Compute the curvature $\kappa : [a, b] \to \mathbb{R}$ in terms of its first and second derivative.
(b) Let $f : [a, b] \to \mathbb{R}$ be a C^2–function; let $\gamma : [a, b] \to \mathbb{R}^2$ be given by $\gamma(t) = (t, f(t))$. Derive formulas for the length of γ and its curvature. Show that the curvature is negative, positive, zero exactly where f is concave, convex or has an inflection point, respectively.
(c) Compute the turning number of γ in (b).

Problem 3
(a) Compute the length and the curvature of the following curves:
$\alpha : [a, b] \to \mathbb{R}^3$; $\alpha(t) = (r \cos t, r \sin t, kt)$ (the helix)
$\beta : (0, \pi) \to \mathbb{R}^2$; $\beta(t) = (\sin t, \cos t + \ln \tan(t/2))$ (tractrix).
$\gamma : (-1, 1) \to \mathbb{R}^2$; $\gamma(t) = (t^2, t^3)$ (semicubic parabola).
$\delta : [0, 2\pi] \to \mathbb{R}^2$; $\delta(t) = (\cos t, \sin(2t))$ (lemniscate of Gerono).
(b) Compute the turning number of δ.