Problem Set 10

Differential Geometry
WS 2019/20

Problems 1 to 3 can be discussed in the tutorial.
You may submit solutions for Problems 4 and 5 until January 15.

Problem 1[Surfaces of revolution]
Let $\gamma = (r, z) : I \to (0, \infty) \times \mathbb{R}$ be either an injective or a simple closed regular plane curve. Define F to be the image of $\Phi : I \times \mathbb{R} \to \mathbb{R}^3$ given by

$$\Phi(t, \theta) = (r(t)\cos\theta, r(t)\sin\theta, z(t))$$

(i) Show that F is a regular surface.
(ii) Compute the first fundamental form of F.
(iii) Determine the Gauss map, the Weingarten map, principal curvatures and corresponding directions, mean and Gaussian curvature in terms of γ and its derivatives.

Problem 2
(i) Recall the parametrizations $\varphi_\pm$ of Problem 2, Set 6. Let p be a non-constant complex polynomial. Define a map $\overline{p} : S^2 \to S^2$ via

$$\overline{p}(x) = \begin{cases} \varphi(p(\varphi_+^{-1}(x))) & \text{for } x \neq (0,0,1) \\ (0,0,1) & \text{else.} \end{cases}$$

Show that $\overline{p}$ is a differentiable map.
(ii) Compute the differential $d_z p$ of the polynomial considered as a differentiable map $p : \mathbb{C} \to \mathbb{C}$. Explain why the set of critical points $C(p) \subset \mathbb{C}$,

$$C(p) := \{z \in \mathbb{C} \mid d_z p(\mathbb{C}) \neq \mathbb{C}\}$$

is finite. Denote by $C_0(p) := \{p(z) \mid z \in C(p), \nexists\, \zeta \in \mathbb{C} \setminus C(p) : p(\zeta) = p(z)\}$. Show that $p(\mathbb{C} \setminus C(p)) = \mathbb{C} \setminus C_0(p)$. Avoid using the fundamental theorem of algebra and conclude it now.
Hint: Show that $p(\mathbb{C} \setminus C(p))$ is open and closed in $\mathbb{C} \setminus C_0(p)$. (i) is not necessary for the argument but can be conveniently used.

Problem 3
Consider the following equivalence relation on the square $[0,1] \times [0,1]$: $(0,t) \sim (1,t)$, $(s,0) \sim (1-s,1)$ and $(s,t) \sim (s,t)$ for all $s, t \in [0,1]$. Show that the quotient space $K^2 := [0,1] \times [0,1]/\sim$ carries a metric for which the projection map $[0,1] \times [0,1] \to K^2$ is continuous and open. i.e. maps open subsets to open subsets and that the metric space K^2 is a 2–dimensional manifold.

Problem 4

Let $p : F \to S^2$ be a local diffeomorphism. Show the lifting property for paths and homotopies of paths as used in class: Let X be any metric space, $\varphi : [0,1] \times X \to S^2$ be a continuous map. Let $\tilde{\varphi}_0 X \to F$ be a continuous map such that $p \circ \tilde{\varphi}_0 = \varphi(0,.)$. Then there exists a unique continuous map $\tilde{\varphi} : [0,1] \times X \to F$ such that $p \circ \tilde{\varphi} = \varphi$ and $\tilde{\varphi}(0,.) = \tilde{\varphi}_0$.

Problem 5

Consider the following equivalence relation on $S^{2n+1} \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$: for $z, z' \in S^{2n+1}$

$$z \sim z' \qquad \Leftrightarrow \qquad z' = \lambda z$$

for some $\lambda \in S^1$. The "quotient space" of equivalence classes is denoted by S^{2n+1}/S^1.

(i) Show that d given by

$$d([z],[z']) = \inf\{\|\zeta - \zeta'\| \mid \zeta \sim z, \zeta' \sim z'\}$$

for equivalence classes $[z],[z'] \in S^{2n+1}/S^1$ defines a metric on the quotient.

(ii) Show that $(S^{2n+1}/S^1, d)$ is a manifold of dimension $2n$.