Problem Set 11

Differential Geometry
WS 2019/20

Problems 1 to 3 can be discussed in the tutorial.
You may submit solutions for Problems 4–6 until January 22.

Problem 1
(i) Check that

$$X(x,y,z) = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}, Y(x,y,z) = \begin{pmatrix} xz \\ yz \\ 1-z^2 \end{pmatrix}$$

define smooth vector fields on $S^2 \subset \mathbb{R}^3$ and express them in stereographic coordinates.
(ii) Let $\alpha \in \mathbb{R}^*$ be a linear form on $\mathbb{R}^3$. Express the smooth one form $\iota^*\alpha = \alpha\mid_{TS^2}$ in spherical coordinates.
(iii) Express the Riemannian metric on $S^2 \subset \mathbb{R}^3$ induced by the Euclidean metric in stereographic coordinates.

Problem 2
(i) Show that

$$G := \{A \in M(n;\mathbb{R}) \mid A^T A = E_n, \det A = 1\}$$

is a submanifold of the space of $n \times n$–matrices $M(n;\mathbb{R}) \cong \mathbb{R}^{n^2}$ and a subgroup of the group. $Gl(n;\mathbb{R})$ of invertible $n \times n$–matrices and determine its dimension. Explain why

$$\mu : G \times G \to G, \quad \mu(g,h) = gh^{-1}$$

is a smooth map.
(ii) Describe the tangent space $T_{E_n}G$. For $g \in G$ let $L_g : G \to G$ denote the smooth map $L_g(h) = gh$. For $v \in T_{E_n}$ define a vector field X on G by

$$X(g) := d_{E_n}L_g(v)$$

Show that for all $g,h \in G$

$$X(gh) = d_h L_g X(h),$$

and that X is differentiable. Moreover, if $\{v_j\} \subset T_{E_n}G$ is a basis, then for the corresponding vector fields $\{X_j(g)\}$ is a basis of T_gG. (iii) Finally let X, Y be the vector fields corresponding to two elements $v, w \in T_{E_n}G$. Show that their Lie bracket $[X,Y]$ is also a vector field corresponding to an element in $T_{E_n}G$. Describe that element.

Problem 3
Let $\gamma : [a,b] \to M$ be a differentiable curve in a differentiable manifold M and V be a vector field along γ. Show that there is a differentiable map $\Gamma : (-\epsilon,\epsilon) \times [a,b] \to M$ such that

$$\frac{\partial}{\partial s}\Big|_{s=0}(\Gamma(s,t)) = V(t).$$

Hint: Prove the statement, when $\gamma([a,b])$ is contained in a coordinate chart. Then cover $[a,b]$ by finite number of intervalls which are mapped to coordinate chart under γ.

Problem 4

For smooth vector fields X, Y, Z and a smooth real function f on a smooth manifold show that

$$[X,Y] = -[Y,X]$$
$$[X,fY] = f[X,Y] + X(f)Y$$
$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

Problem 5

Let (M, g) be a Riemannian manifold, $p, q \in M$ two points,

$$d_g(p,q) := \inf\{\ell_g(\gamma) \mid \gamma : [0,1] \to M \in C^1([0,1], M), \gamma(0) = p, \gamma(1) = q\}.$$

Show that (M, d_g) is a metric space.

Problem 6

Consider the manifold $M := S^{2n+1}/S^1$ from Problem 5 of Problem Set 10. Correction: $S^{2n+1} \subset \mathbb{C}^{n+1}$!

(i) Show that $\{\varphi_j : \mathbb{C}^n \to M\}_{j \in \{1,\dots,n+1\}}$

$$\varphi_j(z_1, ..., z_n) := \frac{1}{\sqrt{1 + \|\mathbf{z}\|^2}}(z_1, ..., z_{j-1}, 1, z_j, ..., z_n)$$

where $\mathbf{z} = (z_1, ..., z_n)$, define a smooth atlas $\{U_j, \varphi_j^{-1}, \mathbb{C}^n\}_j$ of M, where $U_j := \varphi_j(\mathbb{C}^n)$.

(ii) Show that the canonical projection $\pi : S^{2n+1} \to M$, $p(x) = [x]$ where $[x]$ denotes the equivalence class is smooth.

(iii) Show that the tangent space $T_[x]M$ at $[x] \in M$ for a given $x \in S^{2n+1}$ can be identified with the orthogonal complement $(T_x(S^1 \cdot x))^\perp \subset T_x S^{2n+1}$ in $T_x S^{2n+1}$ w.r.t. the scalar product induced by the Euclidean scalar product, where the identification is given by the differential $d_x\pi$. Show that the resulting isomorphism between $(T_x(S^1 \cdot x))^\perp$ and $(T_{\lambda x}(S^1 \cdot x))^\perp$ for any $\lambda \in S^1$ is an isometry.

(iv) Thus the induced scalar product on $(T_x(S^1 \cdot x))^\perp$ defines a scalar product on each tangent space of M. Show that this is a Riemannian metric.

(v) Compute the metric defined in (iii) w.r.t. the coordinate chart $(U_{n+1}, \varphi_{n+1}^{-1})$.