Problem Set 12

Differential Geometry
WS 2019/20

Problems 1 to 4 can be discussed in the tutorial.
You may submit solutions for Problems 5 and 6 until January 29.

Problem 1
(i) Let $\gamma : I \to M$ be a C^2–curve in a manifold M, I is an interval which satisfies the geodesic equation. Prove that its velocity is constant:

$$g_{\gamma_0(t)}(\dot\gamma_0(t), \dot\gamma_0(t)) \equiv \text{const.}$$

(ii) Let $U \subset \mathbb{R}^n$ be an open subset and g be a Riemannian metric on U which is bounded with respect to the Euclidean norm. Suppose $\gamma : I \to U$ is a maximal solution of the geodesic equation for an interval I, i.e. for any solution $\tilde\gamma : \tilde I \to U$ with $I \subset \tilde I$ and $\tilde\gamma|_I = \gamma$ it follows $I = \tilde I$.
Show that if $a := \sup I < \infty$ then there exists a sequence $\{t_n\}_n \subset I$, $t_n \to a$ such that $\{\gamma(t_n)\}_n$ converges in $\mathbb{R}^n \setminus U$. A similar statement holds if $b := \inf I > -\infty$ and does not need to be repeated.

Problem 2
Let (M, g) be a Riemannian manifold, and ∇ its Levi–Civita connection. Let (U, φ, V) be a coordinate chart. Then the Christoffel symbols $\Gamma^k_{ij} : U \to \mathbb{R}$ are defines via

$$\nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} = \sum_{k=1}^n \Gamma^k_{ij} \frac{\partial}{\partial x_k}.$$

for the coordinate vector fields $\frac{\partial}{\partial x_i}$
(i) Show that

$$\Gamma^k_{ij} := \frac{1}{2} \sum_{\ell=1}^n g^{k\ell} \Big(\frac{\partial g_{j\ell}}{\partial x_i} + \frac{\partial g_{\ell i}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_\ell} \Big).$$

(ii) Derive relations between the different Christoffel symbols with permuted indices from the properties of ∇.
(iii) Let $(x_1, ..., x_n)$ and $(\tilde x_1, ..., \tilde x_n)$ be coordinates of two overlapping coordinate charts. Derive a formula for the Christoffel symbols $\tilde\Gamma^k_{ij}$ w.r.t. the second chart in terms of the first Christoffel symbols Γ^k_{ij} and the transition map $\varphi = \varphi(x_1, ..., x_n)$ and its derivatives.

Problem 3
(i) Compute the Christoffel symbols for the Riemannian metric on $S^2 \subset \mathbb{R}^3$ induced by the Euclidean scalar product an $\mathbb{R}^3$ in spherical coordiantes.
(ii) Determine the geodesic equation in these coordinates.
(iii) Conclude that geodesics of the sphere are exactly grand circles.

Problem 4
Let $\gamma_0 : [a,b] \to M$ be a differentiable curve connecting $p = \gamma(a)$ and $q = \gamma(b)$. Show that the energy functional

$$\mathcal{E}_g(\gamma) := \frac{1}{2}\int_a^b g_{\gamma(t)}(\dot\gamma(t), (\dot\gamma(t))dt$$

attains its minimum among all such curves at γ_0 if and only if the length functional

$$\ell_g(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}dt$$

is minimal at γ_0 among all such curves and γ_0 has constant velocity:

$$g_{\gamma_0(t)}(\dot\gamma_0(t), \dot\gamma_0(t)) \equiv \text{const.}$$

Repeat the the part shown in class. For the only-if-part I suggest the following: Assume that there is a curve γ_1 connecting p and q with shorter length. Given $\epsilon > 0$ construct a **regular** differentiable curve γ_2 such that $\ell(\gamma_2) < \ell(\gamma_1) + \epsilon$.

Problem 5
Find a metric covariant derivative on $\mathbb{R}^n$ with respect to the standard Euclidean product which is not torsion free.

Problem 6
(i) Compute the Christoffel symbols for the Riemannian metric on the upper half plane $\mathbb{H} \subset \mathbb{R}^2$ given by

$$g_{(x_1,x_2)} = \frac{1}{x_2^2}\langle\cdot,\cdot\rangle_{\text{standard}}$$

(ii) Determine the geodesic equations in these coordinates.
(iii) Conclude that geodesics of $(\mathbb{H}, g)$ are the half circles with center on $\mathbb{R} \times \{0\}$ and halflines perpendicular to $\mathbb{R} \times \{0\}$. Notice: The computations where sketched in class and there was also a hint how to approach (iii).