Problem Set 13

Differential Geometry
WS 2019/20

Problems 1 to 3 can be discussed in the tutorial.
You may submit solutions for Problems 4 and 5 until February 5.

Problem 1
(i) Let $F \subset \mathbb{R}^3$ be a regular surface. The first fundamental form of F is a Riemannian metric on F. Define for $X \in T_pF$ and a vector field Y in a neighbourhodd of p

$$\nabla_X Y := \mathrm{proj}^{\perp}_{T_pF} X(Y).$$

for the orthogoan projection $\mathrm{proj}^{\perp}_{T_pF} : \mathbb{R}^3 \mapsto T_pF$ and the componentwise directional derivative $X(Y)$ of Y. Show that ∇ is a torsion-free, metric covariant derivative on vector fields on F.
(ii) Using (i) and the geodesic equation involving the Levi-Civita connection ∇, characterize geodesics on F.
(iii) Determine all geodesics of the sphere $S^2 \subset \mathbb{R}^3$.
(iv) Let $Z \subset \mathbb{R}^3$ be a surface of revolution about the z–axis. Call the intersections of a plane containing the z–axis with Z a meridian (actually the connected component). Prove Clairaut's relation: If γ is a geodesic on Z, an ψ the angle it forms with the meridian at some point, while $r > 0$ is the distance from the z–axis at the same point, then $r \sin \psi$ is independent of the point, i.e. a constant. Discuss the converse.
Notice: (i) and (ii) are true or reasonable to ask for an arbitrary submanifolds in $\mathbb{R}^n$ and appropriate versions hold for submanifolds in Riemannian manifolds.

Problem 2
Show Proposition 81 of the lecture: Let $\varphi : N \to M$ be a differentiable map between manifolds, ∇ a covariant derivative on vector fields on M. Then there is unique covariant derivative on vector fields along φ, which assigns for each $p \in N$, a tangent vector $X \in T_pN$ and a vector field Y along φ (defined in a neighbourhood of p, a tangent vector $\nabla^{\varphi}_X Y \in T_{\varphi(p)}M$, which satisfies the following conditions:
(i) $(X, Y) \mapsto \nabla^{\varphi}_X Y$ is linear in X and Y
(ii) Leibniz rule: for a differentiable function f defined on a neighbourhood of p we have
$\nabla^{\varphi}_X (fY) = X(f)Y(\varphi(p)) + f\nabla^{\varphi}_X Y$
(iii) For a vector field Y (defined on a neighborhood of $\varphi(p)$ we have $\nabla^{\varphi}_X (Y \circ \varphi) = \nabla_{f_* X} Y$.
(iv) Moreover, if g is a Riemannian metric on M and ∇ is metric, then ∇^{φ} is allso metric
(v) Finally, if ∇ is torsion free, then ∇^{φ} is torsion free in the following sense: If Z_1, Z_2 are vector fields on N, $\varphi_* Z_i$ define vector fields along φ and we have $\nabla^{\varphi}_{Z_1}(\varphi_* Z_2) - \nabla^{\varphi}_{Z_2}(\varphi_* Z_1) = \varphi_*[Z_1, Z_2]$.
(vi) For coordinates $(z^1, ..., z^n)$ around p in N and $(x^1, ..., x^m)$ around $\varphi(p)$ in M describe the functions $A^j_{\alpha i}$ around p in N for which

$$\nabla^{\varphi}_{\frac{\partial}{\partial z_\alpha}} \frac{\partial}{\partial x_i} = \sum_{\alpha, j} A^j_{\alpha i} \frac{\partial}{\partial x_j}$$

in terms of the Christoffel symbols of ∇ and (derivatives of) φ.

Problem 3
(i) What is the replacement for the first Bianchi identity if ∇ is not torsion-free. Compute the contribution of the torsion T.
(ii) Prove the second Bianchi identity: for all $X, Y, Z \in T_pM$:

$$(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y).$$

Problem 4
Complete the proof that the right hand side in the efinition of the Riemann curvature tensor which assigns to vector fields X, X, Z in a neighborhood of p a vector field

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla[X,Z]Z$$

that vector field at p depends only on X_p, Y_p, Z_p. For this, show that for any differentiable function f on that neighbourhood of p

$$R(fX,Y)Z = R(X,fY)Z = R(X,Y)(fZ) = fR(X,Y)Z.$$

Problem 5
(i) Show that the torsion of a covariant derivative defined on vector fields X, Y on a neighbourhood of p via

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

depends only on X_p, Y_p, Z_p at p. For this, show that for any differentiable function f on that neighbourhood of p

$$T(fX,Y) = T(X,fY) = fT(X,Y)$$

at p.
(ii) Let ∇ be a covariant derivative on vector fields of M. We define the second covariant derivative of a vector field Z on M in direction of $X, Y \in T_pM$ via

$$\nabla^2_{X,Y} := \nabla_X(\nabla_{\tilde{Y}} Z) - \nabla_{\nabla_X \tilde{Y}} Z$$

where $\tilde{Y}$ is a vector field in a neighborhood of p with $\tilde{Y}(p) = Y$. Show that the definition is independent of the extension $\tilde{Y}$ of Y. Show that

$$(X,Y,Z) \mapsto \nabla^2_{X,Y} - \nabla^2_{Y,X} Z$$

for vector fields X, Y, Z defines a $(3,1)$–tensor if and only if ∇ is torsion free. Show that under this condition

$$R(X,Y)Z = \nabla^2_{X,Y} - \nabla^2_{Y,X}.$$

(iii) Let g be a Riemannian metric and ∇ be a covariant derivative on vector fields. Show that ∇ is metric if and only if $\nabla g \equiv 0$ for the induced covariant derivative on $(2,0)$–tensors.