Problem Set 14

Differential Geometry
WS 2019/20

Problems 1 to 3 can be discussed in the tutorial.
You may submit solutions for Problems 4 to 6 until February 12.

Problem 1
Let $f : X \to Y$ be a continous, injective map between metric spaces where X is compact. Prove that f is a homeomprhism onto its image. Show that the assumption on X is necessary by providing counterexamples.

Problem 2
(a) Prove that the Ricci curvature is a symmetric $(2, 0$–tensor.
(b) Compute its covariant derivative.
(c) Show that in local coordinates Gaussian curvature is equal to

$$K = \sum_{i,j,k=1}^{2} g^{jk} R^{i}_{ijk}.$$

Problem 3
Compute the sectional curvature of the sphere $S^n \subset \mathbb{R}^{n+1}$ for $n \geq 2$.

Problem 4
Attempt to prove Gauss' equation (Proposition 94) without using coordinates.

Problem 5
Let $\kappa \in \mathbb{R}$,

$$S_\kappa := \begin{cases} \mathbb{R}^2 & \text{if } \kappa \geq 0 \\ \{(x, y) \mid x_1^2 + x_2^2 < -\frac{4}{\kappa}\} & \text{if } \kappa < 0. \end{cases}$$

Equip S_κ with the Riemannian metric given by

$$g_{ij} := \frac{1}{(1 + \kappa(x^2 + y^2)/4)^2} \delta_{ij}.$$

Compute its Gaussian curvature.

Problem 6
Compute the sectional curvature for $\mathbb{C}P^n := S^{2n+1}/S^1$ (see Problem 6 Problem Set 11).