Problem Set 15

Differential Geometry
WS 2019/20

Problem 1
(i) Describe the exponential maps for the sphere S^2 and the hyperbolic plane $\mathbb{H}^2$ (at a point of your choice).
(ii) Determine the images of disks under these maps without refering to results like the Gauss-Lemma.
(iii) Show that the boundary of these disks (for small enough radius in the case of S^2) are differentiable and perpendicular to the geodesics amanating from the center.
(iv) Do the exponential maps discussed in (i) preserve angles?

Problem 2
Let λ be a differentable one form on a manifold M. Define its differential $d\lambda$ at $p \in M$ to be the anti-symmetric bilinear-form which satisfies for $X, Y \in T_pM$

$$d\lambda(X, Y) = X(\lambda(\tilde{Y})) - Y(\lambda(\tilde{X})) - \lambda_p([\tilde{X}, \tilde{Y}]_p)$$

for vector fields $\tilde{X}, \tilde{Y}$ in a neighboprhood of p with $\tilde{X}_p = X$ and $\tilde{Y}_p = Y$. Show that the right hand side of the identity defines a $(2,0)$–tensor and is, therefore, independent of the choices of $\tilde{X}, \tilde{Y}$. Express $d\lambda$ in terms of λ and its derivatives in a coordinate chart (see lecture).
(2) Anti-symmetric bilinear forms are called 2–forms. Come up with a definition of a 3–form $d\alpha$ for a differentiable 2–form either in coordinates or similar to the above definition.

Problem 3
Sketch the phase diagram near isolated zeros of a vector field in the plane with index $-3, -2, -1, 0, 1, 2$ and 3. The "phase diagram" consists of flow lines corresponding to the vector field.

Problem 4
Let $c : I \to F$ be a differentiable curve in an oriented surface F equipped with a Riemannian metric g with constant velocity, i.e. $g_{c(t)}(\dot{c}(t), \dot{c}(t)) \equiv 1$. Let $\{n(t) \in T_{c(t)}F\}_t$ be the normal field of c such that $\{\dot{c}(t), n(t)\}$ form oriented orthonormal bases of $T_{c(t)}F$. Then

$$k_g(t) := g_{c(t)}(\nabla_{\dot{c}(t)}\dot{c}(t), n(t))$$

is the geodesic curvature of c.
(i) Let $\Delta \subset \mathbb{R}^2$ be a triangle, $U \subset \mathbb{R}^2$ be an open subset, $\Delta \subset U$, g a Riemannian metric on U, α, β, γ be the sizes of the exterior angles at the vertices A, B, C w.r.t. g_A, g_B, g_C. Then

$$\int_\Delta K_g dvol_g = 1 - \int_{\partial\Delta} k_g dt - (\alpha + \beta + \gamma),$$

where the sides of the triangle are arc-length parameztrized by t. Prove Gauss-Bonnet assuming that any surface admits a differentiable triangulation and using that identity. Hint: Consider a vector field on Δ with exactly one zero of index 1 in the interior of Δ and pointing outward at the boundary.
$(ii)^*$ Explain what a surface F with biundary and corners should be (an example being polygons in $\mathbb{R}^2$). Find a formula for the sum of indices of zeros of a vector field on F which points outward at the boundary in terms of K_g, k_g and the exterior angles. Notice that in the case of an overstretched interior angle at a corner (which appears to be non-convex) the exterior angle should be defined to be negative (explain why).