Problem Set 2

Differential Geometry
WS 2019/20

Problem 1
For an open interval $I \subset \mathbb{R}$ let $\gamma : I \to B_R \subset \mathbb{R}^2$ be a regular parametrized curve in the closed disk B_R of radius R, For $t_0 \in I$, $\gamma(t_0) \in \partial B_R$, i.e. γ touches the the boundary of the disk from inside. Show that the absolute value of the curvature of γ at this point is at least $1/R$.

The problems 2 and 4 serve as a guide to repeat and complete arguments in class.

Problem 2 [Lifting-Lemma]
Let $\varphi : I \to S^1$ be a continuous function on an interval $I \subset \mathbb{R}$. Let $t_0 \in I$, and $x_0 \in \mathbb{R}$ such that $\varphi(t_0) = e^{2\pi \mathrm{i} x_0}$. A function $\tilde{\varphi} : I \to \mathbb{R}$ satisfying $\varphi(t) = e^{2\pi \mathrm{i} \tilde{\varphi}(t)}$ for all $t \in I$ and $\tilde{\varphi}(t_0) = x_0$ is called **lift of φ with initial value x_0 at t_0**.
(i) Show that any two such lifts with the same initial value agree. Hint: You can argue similar to Proposition 9 (2).
(ii) Explain why for intervals I_1, I_2 containing t_0, lifts $\tilde{\varphi}_{I_1}$ of $\varphi|_{I_1}$ and $\tilde{\varphi}_{I_2}$ of $\varphi|_{I_2}$ with the given initial value define a lift of $\varphi|_{I_1 \cup I_2}$ with that initial value. Now define the maximal interval $I' \subset I$ with $t_0 \in I'$ for which a lift $\tilde{\varphi}_{I'}$ of $\varphi|_{I'}$ with given initial value exists. Show that $I' = I$ using the arguments from class.
(iii) Let $H : [0,1] \times I \to S^1$ be continuous and $\tilde{H} : [0,1] \times I \to \mathbb{R}$ be a map such that $H(s,t) = e^{2\pi \mathrm{i} \tilde{H}(s,t)}$ for all s, t. Assume that $\tilde{H}(s,.) : I \to \mathbb{R}$ is continuous for all $s \in [0,1]$ and $\tilde{H}(.,t_0) : [0,1] \to \mathbb{R}$ is continuous. Show that then $\tilde{H}$ must be continuous.
(iv) Show that the mapping degrees of two continuous maps $\varphi_0, \varphi_1 : S^1 \to S^1$ which are homotopic agree.

Problem 3
(i) Show that there is no continous map $u : B \to S^1 \subset B^1$ of the closed unit disk to its boundary such that $u|_{S^1} = \mathrm{id}_{S^1}$.
(ii) Show that any continuous map $u : B^2 \to B^2$ admits a fixed point. For that assume the contrary and construct an impossible map as in (i) using the uniquely defined line through x and $u(x)$. Describe the map first geometrically and then by an explicit formula.
(iii) Is the statement in (ii) also true for continuous maps from the open disk to itself?

Problem 4
(i) Show that the map $e : \Delta \to S^1$ defined in the proof of Proposition 10 in class is continuous (see also Christian Bär's book, Proof of Proposition 2.2.10).
(ii) Show that $\delta : [0, L] \to \Delta$ given by $\delta(t) = (t,t)$ is homotopic to $\alpha : [0, L] \to \Delta$ given by

$$\alpha(t) := \begin{cases} (0, 2t) & \text{if } t \in [0, L/2] \\ (2t - L, L) & \text{if } t \in [L/2, L]. \end{cases}$$

Conclude that $e \circ \delta$ is homotopic to $e \circ \alpha$ using this homotopy.
(iii) Show that α is homotopic to $\varphi : t \in [0, L] \mapsto e^{2\pi \mathrm{i} t/L} \in S^1$ by showing that $\alpha|_{[0,L/2]}$ is homotopic to $\varphi|_{[0,L/2]}$ and $\alpha|_{[L/2,L]}$ is homotopic to $\varphi|_{[L/2,L]}$ fixing the boundaries in both cases. Conclude that $\deg(\delta) = 1$.