# Problem Set 3

## Differential Geometry
## WS 2019/20

Problems 1, 2, and 3 can be discussed in class.
You may submit solutions for Problems 4,5 and 6 until November 13.

**Problem 1** [Winding Number]
(i) Draw some interesting closed plane curves, identify the connected regions in its complement and determine the winding number of points in these.
(ii) Let $\gamma : [0,1] \to \mathbb{R}^2$ be a closed plane curve, $p, q \in \mathbb{R}^2 \setminus \gamma$ be two points in its complement which lie in the same connected component, i.e. there is a continuous $\delta : [0,1] \to \mathbb{R}^2 \setminus \gamma$ with $\delta(0) = p$ and $\delta(1) = q$. Show that the winding numbers of $\gamma$ with respect to $p$ and $q$ agree

$$w(\gamma, p) = w(\gamma, q).$$

(iii) Let $h : [0,1] \times [0,1] \to \mathbb{R}^2$ be continuous, $h(s,0) = h(s,1)$ for all $s$ and denote by $\gamma_k : [0,1] \to \mathbb{R}^2$ the closed curve $\gamma_k(t) = h(k,t)$ for $k = 0,1$. Let $p \in \mathbb{R}^2 \setminus im(h)$.

$$w(\gamma_0, p) = w(\gamma_1, p).$$

**Problem 2** [Jordan Curve Theorem]
We will only state this theorem for regular simple closed curves. One can extend the result to piece-wise differentiable simple closed curves without much difficulty. It is true for continuous simple closed curves. There is a short proof (see https://www.maths.ed.ac.uk/ v1ranick/jordan/maehara.pdf) which does not use winding numbers but Brouwer's fixed point theorem in dimension 2 instead (see Problem Set 2, Problem 3 (ii)).

Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a regular simple closed curve ($\gamma$ regular, $L$-periodic and $\gamma|_{[0,L)}$ is injective. Then its complement is the disjoint union of two connected open sets $U, V$ of which one is bounded. The image of $\gamma$ is the topological boundary of either of them.

You may assume that $\gamma$ is arc-length parametrized.
Steps: (1) Show that for $\epsilon > 0$ small enough the map $\Phi : (-\epsilon, \epsilon) \times \mathbb{R} \to \mathbb{R}^2$ given by

$$\Phi(s,t) = \gamma(t) + s n_\gamma(t)$$

is $L$–periodic, a local diffeomorphism and $\Phi|_{(-\epsilon,\epsilon)\times[0,L)}$ is injective and the image is the $\epsilon$–neighbourhood $N_\epsilon$ of $im(\gamma)$:

$$N_\epsilon = \{x \in \mathbb{R}^2 \mid \operatorname{dist}(x, im(\gamma)) < \epsilon\}.$$

Show that $N_\epsilon \setminus \gamma$ has two connected components. It is called the bi-collar neighbourhood of $\gamma$.
(2) The winding number of $\gamma$ w.r.t. points lying in one connected component of $N\epsilon$ differ by $\pm 1$ from those w.r.t. points lying in the other.
(3) Show that each point in $\mathbb{R}^2 \setminus \gamma$ can be connected to exactly one of the two connected components in (2) by a continuous path in $\mathbb{R}^2 \setminus \gamma$ thus identifying two connected components in $\mathbb{R}^2 \setminus \gamma$. Show that one component is unbounded and the winding number of $\gamma$ w.r.t. points in it is zero. Conclude that the other component must be bounded and the corresponding winding number is $\pm 1$.
(4) Show that the image of $\gamma$ is the topological boundary of either of them.

**Problem 3**
Let $\gamma : I \to \mathbb{R}^2$ be a regular plane curve.
(i) Recall the definition of convexity for $\gamma$ from class.
(ii) Prove Lemma 13 from class (Lemma 2.2.14 in Ch. Bär's book): $\gamma$ is convex if and only if for all $t_0, t \in I$

$$\langle \gamma(t) - \gamma(t_0), n_\gamma(t_0) \rangle \geq 0$$

or for all $t_0, t \in I$

$$\langle \gamma(t) - \gamma(t_0), n_\gamma(t_0) \rangle \leq 0$$

You may follow the Argument in Bär's book, carefully explaining it.

**Problem 4**
Give a proof of the slightly stronger statement for Proposition 14: If $\gamma$ is a regular, closed plane curve of class $C^2$ with curvature $\kappa \geq 0$ and turning number $n_\gamma = 1$ then $\gamma$ is convex. We will be very happy to read your suggestions and ask you to present them!

**Problem 5**
(i) Show that a regular plane curve of class $C^k$ can be locally modelled as the graph of a function of class $C^k$ by choosing an appropriate Cartesian coordinate system (or applying an isometry to the plane).
(ii) Complete the final argument of the proof of Proposition 14 (Proposition 2.2.15 in Ch. Bär's book) for the case that $l = 0$.

**Problem 6**
Complete the proof of the Theorem of Whitney and Graustein for the case of regular plane curves with turning number 0 (do not try to follow the hint given in class):
(i) Show that any closed regular plane curve $\gamma : [0,1] \to \mathbb{R}^2$ is regular homotopic to another closed regular plane curve $\hat{\gamma} : [0, L] \to \mathbb{R}^2$ with curvature $\kappa(t) > 0$ for $t$ close to 0 and 1.
(ii) Assume now w.l.o.g. that $\gamma_0$ and $\gamma_1$ satisfy the latter condition. Prove that for the homotopy $h : [0,1] \times [0,1] \to \mathbb{R}^2$ $h(s, .)$ is never constant hence concluding the proof of the theorem.