Problem Set 4

Differential Geometry
WS 2019/20

Problems 1 and 2 can be discussed in class.
You may submit solutions for Problems 3 and 4 until November 20.

Problem 1
(a) Let $\gamma : [a, b] \to \mathbb{R}^3$ be a regular parametrized curve (not necessarily arc-length parametrized) with nowhere vanishing curvature. Show that $\ddot{\gamma}$ is perpendicular to the binormal.
(b) Derive formulas for curvature, torsion and the reference frame for γ as in (a).
(c) Compute curvature, torsion and the reference frame for the helix.
(d) Show Frenet's formulas.

Problem 2 [Spherical geometry]
(a) Show that for $P, Q \in S^2$ the shorter arc of the circle with center 0 and radius 1 is the shortest regular curve on S^2 connecting P and Q.
(b) For $P, Q \in S^2$ we define

$$d^{S^2}(P, Q) = \measuredangle(POQ).$$

Show that d^{S^2} defines a metric on S^2.
(b) For a curve $\gamma : [a, b] \to S^2$ show that

$$\ell(\gamma) = \sup\{\sum_{i=1}^{m} d^{S^2}(\gamma(t_i), \gamma(t_{i-1}) \mid a = t_0 < t_1 < ... < t_m = b\}.$$

where $\ell(\gamma)$ is the length of γ considered as a space curve.

Problem 3
Show that the total angle of a closed polygon is equal to 2π if and only if the polygon is planar, simple and convex.

Problem 4
Denote by $\mathbb{H} : \{(x_1, x_2) \mid x_2 > 0\} \subset \mathbb{R}^2$ the (open) upper half plane. The hyperbolic length $\ell^{\mathbb{H}}(\gamma)$ of a C^1–curve $\gamma : [a, b] \to \mathbb{H}$ is defined to be

$$\ell^{\mathbb{H}}(\gamma) = \int_a^b \frac{1}{x_2(t)} \|\dot{\gamma}(t)\| dt$$

where $\gamma(t) = (x_1(t), x_2(t))$ are the components and $\|.\|$ denotes the euclidean norm of $\mathbb{R}^2$.
(a) Explain that ℓ does not change under reparametrization of γ.
(b) Let $P = (x_1, x_2), Q = (y_1, y_2) \in \mathbb{H}$ be two points in $\mathbb{H}$. Show that the shortest C^1–curve in $\mathbb{H}$ joining P and Q is given by the euclidean segment PQ if $x_1 = y_1$ or the segment between P and Q of the unique euclidean circle containing P and Q whose center lies on the x_1–axis if $x_1 \neq y_1$.
(c) Define $d^{\mathbb{H}}(P, Q)$ to be the shortest hyperbolic length of a C^1-curve in $\mathbb{H}$ connecting $P, Q \in \mathbb{H}$. Show that this defines a metric on $\mathbb{H}$. Show that it is complete.
(d) Show that for a C^1–curve $\gamma : [a, b] \to \mathbb{H}$

$$\ell^{\mathbb{H}}(\gamma) = \sup\{\sum_{i=1}^{m} d^{\mathbb{H}}(\gamma(t_i), \gamma(t_{i-1}) \mid a = t_0 < t_1 < ... < t_m = b\}.$$

(e) What should the measure of an angle in $\mathbb{H}$ analogous to a euclidean angle be? What are convex sets and therefore convex polygons of hyperbolic segments? Explain that the total angle of a simple closed convex hyperbolic polygon is bigger that 2π on one example. Can you prove it to be true in general?