Problem Set 5

Differential Geometry
WS 2019/20

Problems 1–3 can be discussed in class.
You may submit solutions for Problems 4 and 5 until November 27.

Problem 1
Let $\gamma : [a, b] \to \mathbb{R}^3$ be a C^2 map. Show that for all $s, t \in [a, b]$ $s \neq t$ we have

$$\|\dot{\gamma}(s) - \frac{\gamma(s) - \gamma(t)}{s - t}\| \leq \|\ddot{\gamma}\|_{C^0}(s - t)^2.$$

where $\|.\|$ denotes the euclidean and $\|.\|_{C^0}$ denotes the corresponding supremum norm.

Problem 2
(ii) Let β be an arc on a grand circle of the unit sphere S^2. Show that the set

$$\{x \in S^2 \mid x^{\perp} \cap \beta \neq \emptyset\}$$

is the union of two diametrical spherical two-gons whose interior angles are equal to the length $\ell(\beta)$.

Problem 3
Let $\gamma_1\gamma_2,\ [a, b] \to \mathbb{R}^3$ be continuous space curves, $x \in S^2$ $t \in (a, b)$ and $\gamma_1(a) = \gamma_2(a), \gamma_1(b) = \gamma_2(b), \gamma_1(t) = \gamma_2(t)$. Assume that $s \in [a, b] \to \langle \gamma_1(s), x\rangle$ admits a local maximum at $t \in (a, b)$. Show that $s \in [a, b] \to \langle \gamma(s), x\rangle$ has a local maximum in the open interval (a, b). Identify the two situations in the proof of Proposition 30 where this property has been exploited.

Problem 4
(i) Determine the function $\mu(\gamma, .) : S^2 \to \mathbb{N} \cup \{\infty\}$ for the unit circle in the plane $\mathbb{R}^2 \times \{0\}$ and check the validity of Proposition 30.
(ii) Show for the bridge number $\mu(\gamma, x)$ for a continuous periodic curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ and $x \in S^2$ as defined in class that for all $x \in S^2$ we have $\mu(\gamma, x) = \mu(\gamma, -x)$.

Problem 5
(i) Let γ be a simple closed regular space curve of class at least C^2. Show that for the total curvature $\kappa(\gamma) = 2\pi$ if and only if γ is a convex plane curve (2nd part of Fenchel's theorem).
(ii) Prove for a polygon $\mathcal{P}$ in $\mathbb{R}^3$ that if the total angle $\kappa(\mathcal{P}) = 2\pi$ then $\mathcal{P}$ is a convex plane polygon.
(iii) Use this statement to show that "simpleness" can be dropped in the "if"–direction of (i). Why can it not be dropped in the "only if"–direction?