Problem Set 6

Differential Geometry
WS 2019/20

Problems 1 to 4 can be discussed in class.
You may submit solutions for Problems 5 and 6 until December 4.

Problem 1
Check whether the homotopies described in class which deform a simple closed, regular space curve of total curvature less than 4π into a plane circle are regular isotopies, i.e. all curves of the family are simple closed, regular space curves. Adapt them to fulfil that property if necessary.

Problem 2
(i) Show that the maps $\varphi_\pm : \mathbb{R}^2 \to S^2$ given by

$$\varphi_\pm(x, y) = (2x/(x^2 + y^2 + 1), 2y/(x^2 + y^2 + 1), \pm(x^2 + y^2 - 1)/(x^2 + y^2 + 1))$$

are regular parametrizations of the 2–sphere.
(ii) Compute the transition map $\varphi_-^{-1} \circ \varphi_+$. What is this map geometrically?

Problem 3
Let A be a 3×3–matrix for which $A^T A = E_3$ – a so-called special orthogonal matrix. The set of such is denoted by $SO(3)$ and forms a group.
Show that $x \in S^2 \to Ax \in S^2$ defines a diffeomorphism.

Problem 4
(i) Show that the double cone $C \subset \mathbb{R}^3$

$$C := \{(x, y, z) \mid x^2 + y^2 - z^2 = 0\}$$

is not a regular surface. Hint: Assume that $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a regular parametrization, $0 \in U$ and $\varphi(0) = 0$. There have to be two points $p, q \in U$ such that the z–coordinates of $\varphi(p)$ and $\varphi(q)$ have different signs. Consider $\varphi \circ \gamma$ for an injective curve $\varphi : (-\epsilon, \epsilon) \to U$ connecting p and q passing through 0 (see Ch. Bär's book if necessary)
(ii) (i) would also follow from the following statement: C is not an immersed surface. Prove the statement.
(iii) Show that

$$C_+ := C \cap (\mathbb{R}^2 \times [0, \infty)) \subset \mathbb{R}^3$$

is not a regular surface either.

Problem 5
(i) Let $R > r > 0$ be fixed real numbers. Show that the following subset of $\mathbb{R}^3$ is a regular surface:

$$T^2 := \{(x, y, z) \mid (x^2 + y^2 + z^2 + R^2 - r^2)^2 - 4R^2(x^2 + y^2) = 0\}$$

(ii) Cover T^2 by parametrizations (4 will conveniently do).
(iii) What does this surface look like?
Hint: It is a set with a rotational symmetry about the z–axis.

Problem 6
(i) Show that the following two subsets of $\mathbb{R}^2$ are not diffeomorphic:

$$A := [0, \infty) \times \{0\} \cup \{0\} \times [0, \infty)$$

and

$$B := \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^3 = 0\}.$$

Sketch the sets to get an idea how to approach the problem.
(ii) Show that $F : (0, 3\pi/2) \times \mathbb{R} \to \mathbb{R}^3$ given by

$$F(x, y) = (\cos x, \sin(2x), y)$$

is injective and locally an embedding (i.e. defines an immersed surface), but not a surface patch (i.e. the image is not a regular surface). Draw a sketch of an intersection with the plane $\mathbb{R}^2 \times \{y\}$ for any $y \in \mathbb{R}$.