Problem Set 9

Differential Geometry
WS 2019/20

Problems 1 to 4 can be discussed in the tutorial.
You may submit solutions for Problems 5 and 6 until January 8.

Problem 1
Prove the following statement: Let $F \subset \mathbb{R}^3$ be a regular oriented surface, $N : F \to S^2$ be its Gauss map. Let $\gamma : I \to F$ be an arc-length parametrized curve. Then γ is a line of curvature if and if there is a function $\lambda : I \to \mathbb{R}$ satisfying

$$\frac{d}{dt} N(\gamma(t)) = \lambda(t)\dot{\gamma}(t)$$

for all $t \in I$.

Problem 2
Prove the following formula of Euler: Let $X_1, X_2 \in T_pF$ be eigenvectors of the Weingartenmap of a regular surface $F \subset \mathbb{R}^3$, $p \in F$, $\|X_i\| = 1$ for $i = 1, 2$ and $\langle X_1, X_2\rangle = 0$ (Why is this possible?). Denote by κ_1, κ_2 be the corresponding eigenvalues. Consider a vector of unit length $X \in T_pF$. Then $X := \lambda_1 X_1 + \lambda_2 X_2$ with $\lambda_1^2 + \lambda_2^2 = 1$. Show that

$$II_p(X, X) = \lambda_1^2\kappa_1 + \lambda_2^2\kappa_2.$$

Conclude that the principal curvatures are the extremal values of

$$\{II_p(X, X) \mid X \in T_pF, \|X\| = 1\}.$$

Problem 3
Prove Lemma 52: Any regular surface $F \subset \mathbb{R}^3$ can be locally modelled around a apoint as the graph of a differentiable function over the tangent space at this point: If $N(p)$ is a normal at p, then there exist open neighbourhoods $U \subset T_pF \cong \mathbb{R}^2$ of 0, $V \subset \mathbb{R}^3$ of p and a differentiable map $h : U \to \mathbb{R}$, such that

$$F \cap V = \{p + x + h(x)N(p) \mid x \in U\}.$$

Hint: In the lemma we stated that the orthogonal projection to T_pF restricted to $V \cap F$ is a diffeomorphism onto its image form which the claim follows (how?). Als note that in this formulation T_pF is considered to be a vector subspace of $\mathbb{R}^3$ rather than an affine subspace parrallel to it passing through p. Hence p had to be added.
This problem may not be discussed in the tutorial since it is a result from Analysis II (implicit function theorem etc.) You are expected to know how to solve it, so if in doubt, please ask.

Problem 4
Let $F \subset \mathbb{R}^3$ be a regular surface, $p \in F$. Show:
(i) If $K(p) > 0$ or a sufficiently small neighbourhood $V \subset \mathbb{R}^3$ of p, $F \cap V$ lies on one side of the affine tangent plane $p + T_pF$ (a side of the hyperplane is given by

$$\{x \in \mathbb{R}^3 \mid \langle x - p, n\rangle \geq 0\}$$

for a normal $n \neq 0$ to F at p.
(ii) If $K(p) < 0$ than for any neighbourhood $V \subset \mathbb{R}^3$ of p $F \cap V$ lies on both sides of the affine tangent plane $p + T_pF$.
(iii) Explain why either of the two statement is false in the case $K(p) = 0$.

Problem 5
Let $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ be a diffentiable function (at least C^2) for which $0 \in \mathbb{R}$ is a regular value, i.e. for all $x \in f^{-1}(0)$, $d_x f \neq 0$. Explain why $F := f^{-1}(0) \subset \mathbb{R}^3$ is a regular surface. Compute second fundamental form and Weingarten map in terms of f and its derivatives. Derive formulas for its Gaussian and its mean curvature.

Problem 6
Let $F \subset \mathbb{R}^3$ be a compact surface. Prove the following statements:
(i) The Gauss map is surjective.
(ii) If the Gauss map is injective then $K \geq 0$ everywhere.
(iii) Show that the restriction of the Gauss map to the set $\{p \in F \mid K(p) \geq 0\}$ is surjective.
(iv) Show that the restriction of the Gauss map to the set $\{p \in F \mid K(p) > 0\}$ is a local diffeomorphism. Hint: Problem 5 of Problem Set 8 could be helpful.