1 Curves in $\mathbb{R}^2$ and $\mathbb{R}^3$

1. (easy) What is a parametrization by arclength of a regular curve? Show the existence. Define curvature of a plane curve.
2. (easy) Let $\gamma : [a, b] \to \mathbb{R}^2$ be a regular smooth curve. Compute its curvature $\kappa : [a, b] \to \mathbb{R}$ in terms of $\dot{\gamma}$, $\ddot{\gamma}$.
3. (medium) Let $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^2$ be a regular smooth curve such that $\gamma(0) = (R, 0)$ and $\gamma((-\epsilon, \epsilon)) \subset \{z \in \mathbb{C} \mid |z| < 1\}$. Show that $\kappa(0) \geq \frac{1}{R}$.
4. (medium) Define the turning number of a plane curve and discuss its relation to the curvature of the curve.
5. (hard) Define the degree of a continuous map $\varphi : S^1 \to S^1$ and prove Brouwer's fixed point theorem in dimension 2 (Problem 3, Problem Set 2)
6. (hard) Discuss the relation of curvature and convexity of a closed plane curve.
7. (medium) Formulate and prove the isomperimetric inequality in Euclidean plane.
8. (hard) Classify the isotopy classes of immersed closed, connected curves in the plane (Whitney–Graustein).
9. (medium) Define curvature and torsion of a space curve. Discuss how (and when) they determine the curve.
10. (easy) Define the total angle of a space curve and explain its relation to curvature of the curve.
11. (hard) Define the bridge number of a space curve and discuss its relation to the curvature of the curve.
12. (hard) Show Fenchel's theorem (a sharp lower bound on total curvature of a regular space curve with implication if equality is attained).
13. (hard) Explain the Theorem of Fáry and Milnor and the idea of its proof.

2 Surfaces in $\mathbb{R}^3$

1. (easy) What is a regular surface in $\mathbb{R}^3$? Discuss different characterizations. Define its tangent space
2. (easy) Compute the formula for the (inverse of the) stereographic projection $\varphi : \mathbb{R}^2 \to S^2$ from the north pole of the unit sphere in $\mathbb{R}^3$.
3. (easy) Define the first fundamental form of a regular surface and explain how its representation matrix changes under coordinate changes. Explain how this leads to the idea of Riemannian metrics and Riemann tensor.
4. (easy) How can the length of a curve on a surface be computed in local coordinates. Give a definiton of the angle between tangent vectors in terms of the first fundamental form or the Riemann tensor.

5. (easy) Compute the first fundamental form of $S^2(R) \subset \mathbb{R}^3$ in the spherical coordinates given by $\varphi(\theta, \phi) = (R \sin\phi \cos\theta, R \sin\phi \sin\theta, R \cos\phi)$.

6. (easy) Define orientation of a regular surface. Define its normal, the Gauss- and the Weingarten map. Define the second fundamental form.

7. (medium) Compute the second fundamental form of a surface given as the graph of a smooth function at a critical point of that function.

8. (easy) Define principal curvature directions, mean and Gaussian curvature.

9. (hard) Explain their geometric meaning. Show that for any compact regular surface without boundary there is a point with positive Gaussian curvature.

10. (medium) What is the normal curvature of a curve in a regular surface F. What is its relation to the second fundamental form of F (Meusnier's formuila)?How can $II_p(X, X)$ be expressed by principal curvature and normal curvature (Euler's formula) ?

11. (medium) Compute the Gaussian curvature K of the saddle $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 - y^2\}$ at $(0, 0, 0)$. Explain why it is negative.

12. (medium) Let $K \in \mathbb{R}^3$ be a closed surface. Prove that the Gauss map $N : K \to S^2$ is surjective.

3 Riemannian Geometry

1. (easy) Define the notion of a differentiable manifold. Discuss examples.

2. (medium) What is a Riemannian manifold? Prove that a Riemannian metric defines a metric?

3. (medium) Let G be a n–dimensional Lie group. Explain why we can find vector fields $X_1, \dots X_n$ such that $X_1(p), \dots X_n(p)$ form a basis of T_pG for every $p \in G$.

4. (easy) Let (M, g) be a Riemannian manifold and ∇ be its Levi–Civita connection. Explain what are the properties that define ∇ uniquely.

5. (easy) Let (M, g) be a Riemannian manifold and let ∇ be a g–compatible connection. Prove that the curves γ which solve $\nabla_{\dot\gamma}\dot\gamma = 0$ have constant speed.

6. (medium) Derive the formula for critical points of the energy functional on paths $\gamma : [a, b] \to M$ connecting two fixed points of the manifold.

7. (medium) Show that if γ is of minimal length among all curves of constant velocity then it is a minimum of the energy functional.

8. (hard) Show that if γ is a minimum of the energy functional then it is of minimal length among all differentiable curves connecting the tow points.

9. (medium) Write down the definition of the Christoffel symbols of ∇. Compute them in local coordinates in terms of g.

10. (medium) Let $M \in \mathbb{R}^3$ be a hypersurface. Let g be the first fundamental form of M. Let $\nabla : T_pM \times \mathrm{Vect}(M) \to T_pM$ at $p \in M$ be defined by

$$\nabla_X Y := \pi^T(X(Y)),$$

where $\pi^T : T_p\mathbb{R}^3 \to T_pM$ is the projection to the tangent space of T_pM, $X(Y)$ denotes as usual the derivavtive at $p \in M$ in direction of $X \in T_pM$ of each component of the vector field Y around p. Show that ∇ is the Levi–Civita connection of g.

11. (hard) Define the geodesic curvature $\kappa_g : \mathrm{Im}\gamma \to \mathbb{R}$ for a geodesic γ. State the Gauss–Bonnet theorem and give one example that verifies it. Give an idea how one can go about proving this theorem.

12. (easy) Define the exponential map $\exp_p$ of a Riemannian manifold M. Explain why it is a diffeomeorphism in a neighborhood of $0 \in T_pM$. What are normal coordinates? How does the Riemann metric tensor look like in normal coordinates.

13. (hard) Formulate and prove Gauss' Lemma.

14. (hard) Express Gauss curvature in terms of the metric tensor in normal coordinates for a surface.

15. (medium) Explain why the two topologies (open sets) the one originally given for a manifold and the one induced by the metric d_g of a Riemannian metric g agree.

16. (medium) Formulate the Theorem of Hopf-Rinow. Give some examples of complete and incomplete Riemannian manifolds.

17. (hard) Sketch the ideas behind the proof of the Theorem of Hopf-Rinow.