Geodesics, exponential map and completeness in Riemannian geometry

K. Mohnke

Abstract

This is s short script of the classes on the Gauss–Lemma and the Hopf–Rinow-Theorem. The notes are supplementary material. The pictures were presented in class. If there are typos I hope they can be corrected using your notes.

1 Exponential map and normal coordinates

Let (M, g) be an n–dimensional Riemannian manifold, $p \in M, X \in T_pM$. We denote by $\gamma_X : (a, b) \to M$, $\gamma_X(0) = p, \dot{\gamma}_X(0) = X$ the unique geodesic which is a solution of the 2nd order ordinary differential equation

$$\nabla^{\gamma}_{\frac{\partial}{\partial t}} \dot{\gamma} = 0.$$

Then for any $\lambda > 0$ the geodesic $\gamma_{\lambda X} : (a/\lambda, b/\lambda) \to M$ satisfies

$$\gamma_{\lambda X}(t) = \gamma_X(\lambda t).$$

Denote by the open interval $I_X \subset \mathbb{R}$ the maximal solution for given initial values p, X $(0 \in I_X)$, then for $\lambda \neq 0$ $I_{\lambda X} = I_X/\lambda$. As usual, we assume that g is sufficiently differentiable. Then by general theory of ODE, we know that γ_X depends differentiably on p and X. In particular, for $t \in I_X$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $X' \in T_pM$ with $\|X - X'\|_{g_p} < \delta$ and all $t' \in \mathbb{R}$ with $|t - t'| < \varepsilon$, $t' \in I_{X'}$ and

$$(t', X') \in (t - \varepsilon, t + \varepsilon) \times \{X' \in T_pM | X - X' < \delta\} \mapsto \gamma_{X'}(t') \in M$$

is a differentiable map. Consequently

$$V_p := \{X \in T_pM \mid 1 \in I_X\} \subset T_pM$$

is an open subset which is **starshaped** with respect to 0.

Proposition 103: (1) $\exp_p : V_p \to M$ defined by

$$\exp_p(X) := \gamma_X(1)$$

is a differentiable map. With the identification $T_0V_p = T_pM$ and $T_{\exp_p(0)}M = T_p$ (since $\exp_p(0) = p$ we have

$$d_0 \exp_p = \mathrm{id}_{T_pM}$$

(2) There exists a $r_p > 0$ such that $B(p, r_p) := \exp_p |_{B_{T_pM,g_p}}(0, r_p)$ is a diffeomorphism onto its image. Choose an orthonormal basis $\{X_1, ..., X_n\}$ of (T_pM, g_p). Then the parametrization

$$(x_1, ..., x_n) \in B^n(r_p) \subset \mathbb{R}^n \mapsto \exp_p(x_1X_1 + ... + x_nX_n) \in M$$

defined on the euclidean ball define a coordinate chart around p, called **normal coordinate chart** while $(x_1, ..., x_n)$ are called normal coordinates.

(3) With respect to normal coordinates the Riemann tensor satisfies for all $i, j, k \in \{1, ..., n\}$

$$g_{ij}(0) = \delta_{ij} \quad \frac{\partial g_{ij}}{\partial x_k}(0) = 0.$$

Proof: (1) The differentiabilty was discussed prior to the proposition. Now

$$d_0 \exp_p(X) = \frac{d}{dt}|_{t=0} \exp_p(tX) = \frac{d}{dt}\gamma_{tX}(1) = \frac{d}{dt}\gamma_X(t) = X$$

by definition of γ_X.

(2) Since $d_0 \exp_p = \mathrm{id}_{T_pM}$ is invertible by the inverse function theorem there exists an open neighborhood of 0 which can be assumed to be a euclidean ball, which satisfies the claim.

(3) $g_{ij}(0) = \delta_{ij}$ follows directly from $d_0 \exp_p = \mathrm{id}_{T_pM}$ and the orthonormality of $\{X_i\}_i$. Now define the geodesics $\gamma_{ij}(t) := \exp_p(t(X_i + X_j))$, $\gamma_i(t) \exp_p(tX_i)$ for sufficiantly small t. Then

$$0 = (\nabla^{\gamma_{ij}}_{\frac{\partial}{\partial t}} \dot{\gamma}_{ij})(t) = \Big(\nabla_{\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j}} (\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j})\Big)(\gamma_{ij}(t).$$

For the Levi-Citvita connection ∇. Similarly,

$$\big(\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_i}\big)(\gamma_i(t) = 0 \quad \big(\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_j}\big)(\gamma_j(t) = 0$$

At $t = 0$ these three identities amount to

$$0 = \Big(\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} + \nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i} + \nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_i}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_j}\Big)(0)$$

and using that ∇ is torsion–free we obtain for all i, j

$$\Big(\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}\Big)(0) = 0.$$

Finally, we use that ∇ is metric to obtain

$$\frac{\partial g_{ij}}{\partial x_k}(0) = \frac{\partial}{\partial x_k}g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})(0) = g(\nabla_{\frac{\partial}{\partial x_k}}\frac{\partial}{\partial x_i}(0), \frac{\partial}{\partial x_j(0)}) + g(\frac{\partial}{\partial x_i}(0), \nabla_{\frac{\partial}{\partial x_k}}\frac{\partial}{\partial x_j})(0)) = 0 \quad \Box$$

Corollary: For all i, j, k we have in normal coordinates $\Gamma^k_{ij}(0) = 0$.

The follwoing observation by C.F. Gauß is crucial in proving local minimality of geodesics.

Theorem 104 [Gauss–Lemma]: Let (M, g) be a Riemannian manifold of dimension n, $p \in M$, $X \in V_p \subset T_pM$. Then for any other $Y \in T_pM$ we have

$$g_{\exp_p(X)}(d_X \exp_p(X), d_X \exp_p(Y)) = g_p(X, Y), \tag{1}$$

where we have, once again identified $T_X(T_pM) = T_pM$.

In particular, if $\tilde{\delta} : (-\varepsilon, \varepsilon) \to V_p$ satisfies $\tilde{\delta}\|_{g_p} \equiv r$, then for $\delta := \exp_p \circ \tilde{(\delta)}$

$$\dot{\delta}(0) \perp \dot{\gamma}_X(1).$$

Remark: Equation (1) does NOT mean, that $\exp_p$ preserves all angles!

Proof: To shorten notation we introduce $\varphi : (-\varepsilon, \varepsilon) \times [0, 1| \to M$ fgiven by

$$\varphi(s, t) := \exp_p(t(X + sY))$$

for $\varepsilon > 0$ sufficiently small. Then

$$d_X \exp_p(X) = \dot{\gamma}_X(t) = \frac{\partial \varphi}{\partial t}(0, 1)$$

$$d_X \exp_p(Y) = \frac{\partial}{\partial s}\exp_p(X + sY) = \frac{\partial \varphi}{\partial t}(0, 1).$$

Hence we need to show

$$g_{\exp_p(X)}(\frac{\partial\varphi}{\partial t}(0,1), \frac{\partial\varphi}{\partial t}(0,1) = g_p(X,Y).$$

Since $t \mapsto \varphi(s,t)$ defines a geodesic for all s we have

$$\nabla^{\varphi}_{\frac{\partial}{\partial t}} \frac{\partial\varphi}{\partial t}(s,t) = 0$$

and

$$\|\frac{\partial\varphi}{\partial t}(s,t)\|_{g_{\varphi(s,t)}} = \|\frac{\partial\varphi}{\partial t}(s,t)\|_{g_{\varphi(s,0)}} = \|X + sY\|_p$$

for all s, t Using this and that ∇ is metric and torsion free we compute

$$\begin{aligned}
\frac{\partial}{\partial t} g(\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial s}) &= g(\nabla^{\varphi}_{\frac{\partial}{\partial t}} \frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial s}) + g(\frac{\partial\varphi}{\partial t}, \nabla^{\varphi}_{\frac{\partial}{\partial t}} \frac{\partial\varphi}{\partial s}) \\
&= g(\frac{\partial\varphi}{\partial t}, \nabla^{\varphi}_{\frac{\partial}{\partial s}} \frac{\partial\varphi}{\partial t}) \\
&= \frac{1}{2}\frac{\partial}{\partial s} g((\frac{\partial\varphi}{\partial t}, (\frac{\partial\varphi}{\partial t} \\
&= \frac{1}{2}\frac{\partial}{\partial s} g_p(X + sY, X + sY) \\
&= s g_p(Y,Y) + g_p(X,Y).
\end{aligned}$$

Summing up we obtain

$$\frac{\partial}{\partial t} g(\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial s})(0,t) = g_p(X,Y).$$

Together with

$$g(\frac{\partial\varphi}{\partial t}(0,0), \frac{\partial\varphi}{\partial s}(0,0)) = g_p(X,0) = 0$$

and integrating over t we obtain the desired identity. □

Specializing to surfaces $\dim M = 2$ Gauß' Lemma yields the following. For a othonormal basis $\{X_1, X_2\} \subset T_pM$ we denote by $F : (0, \mathbf{r}_p) \times \mathbb{R} \to M$

$$F(r,\varphi) := \exp_p(r(\cos\varphi X_1 + \sin\varphi X_2))$$

the so-called geodesic normal coordinates. With respect to these coordinates the Riemann tensor has the form

$$g(r,\varphi) =: \begin{pmatrix} g_{rr} & g_{r\varphi} \\ g_{\varphi r} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & f^2(r,\varphi) \end{pmatrix}$$

for a differentiable function $f : (0, \mathbf{r}_p) \times \mathbb{R} \to (0, \infty)$ satisfying

$$\lim_{r \to 0} f(r, \varphi) = 0 \quad \lim_{r \to 0} \frac{\partial f}{\partial r} = 1.$$

Indeed, the radial vector field is perpendicular to the angular vector field on T_pM by the choice of X_1, X_2, hence the diagonal form of g is a consequence of Theorem 104. Since $r \mapsto F(r, \varphi)$ defines a geodesic for all φ the left upper entry follows from $\| \cos \varphi X_1 + \sin \varphi X_2 \| = 1$. Finally,

$$\begin{aligned} \frac{f^2(r, \varphi)}{r^2} &= g_{F(r,\varphi)}(d_{r,\varphi} F(-\sin \varphi X_1 + \cos \varphi X_2, -\sin \varphi X_1 + \cos \varphi X_2) \\ &= \sin^2 \varphi g_{11} + \cos^2 \varphi g_{22} + 2 \sin \varphi \cos \varphi g_{12} \end{aligned}$$

where g_{ij} are the coefficients of the Riemann tensor w.r.t. the normal coordinates defined by $\{X_1, X_2\}$. Since g_{ij} is continuous and $g_{ij}(0) = \delta_{ij}$ we obtain

$$\lim_{r \to 0} f(r, \varphi) = \lim_{r \to 0} \sqrt{\sin^2 \varphi g_{11} + \cos^2 \varphi g_{22} + 2 \sin \varphi \cos \varphi g_{12}} = 0$$

and

$$\lim_{r \to 0} \frac{\partial f}{\partial r} = \lim_{r \to 0} \frac{f(r, \varphi)}{r} = \lim_{r \to 0} \sqrt{\sin^2 \varphi g_{11} + \cos^2 \varphi g_{22} + 2 \sin \varphi \cos \varphi g_{12}} = 1.$$

For instance one deduces for the Gaussian curvature

$$K(r, \varphi) = -\frac{1}{f(r, \varphi)} \frac{\partial^2 f}{\partial r^2}(r, \varphi).$$

Corollary: (1) Let M_1, M_2 be surfaces with a Riemannian metric which have the same **constant** Gaussian curvature K, then they are locally isometric. (2) If $B(p, r) := \exp_p(B^2(r))$ for $r < r_p$ and normal coordinates. Then

$$\int_{B(p,r)} K d\mathrm{vol}_g = 2\pi \int_0^{2\pi} \frac{\partial f}{\partial r} d\varphi.$$

Proof: (1) f_1, f_2 are determined by K. Hence the isometry is provided by $\exp_{p_2}^{M_2} \circ (\exp_{p_1}^{M_1})^{-1} : B^1(p_1, r) \to B^2(p_2, r)$.

(2) We have $K d\mathrm{vol}_g = f dr d\varphi$ and thus

$$\int_{B(p,r)} K d\mathrm{vol}_g = -\int_0^r \int_0^{2\pi} \frac{\partial^2 f}{\partial r^2} d\varphi dr$$
$$= -\int_0^{2\pi} \left[\frac{\partial f}{\partial r}(s,\varphi)\right]\Big|_{s=0}^r d\varphi$$
$$= -\int_0^{2\pi} (\frac{\partial f}{\partial r}(r,\varphi) - 1) d\varphi. \quad \square$$

Another corollary of Theorem 104 is the following

<u>Lemma 105:</u> Similar to previous notation denote by $\overline{B(p,r)} = \exp_p(\overline{B_{T_pM,g_p}(r)})$ the image of the closed euclidean ball, where $r < r_p$, s.t. $\exp_p|_{B_{T_pM,g_p}(r)}$ is a diffeomorphism onto its image. Let $\gamma : [0,T] \to \overline{B(p,r)}$ be a piecewise differentiable curve s.t. $\gamma(0) = p, \gamma(T) in S(p,r) = \partial B(p,r)$. Then the length $\ell(\gamma) \geq r$ with equality if and only if $\gamma(t) = \exp_p(tX)$ with $X \in T_pM$, $\|X\|_{g_p} = 1$.

Proof: There exist $r : [0;T] \to [0,r], \eta : [0,T] \to S_p := \{X \in T_pM \mid \|X\| = 1\}$, such that

$$\gamma(t) = \exp_p(r(t)\eta(t))$$

where r is piecewise differentiable and continuous and uniquely determined and η is piecewise differentiable but possibly not continuous at t with $r(t) = 0$. Then

$$\dot\gamma(t) = d_{\gamma(t)} \exp_p(\dot r(t)\eta(t)) + d_{\gamma(t)} \exp_p(r(t)\dot\eta(t)). \tag{2}$$

Since $\|\eta(t)\| \equiv 1$ $g_p(\dot\eta(t), \eta(t) = 0$ whereever $\dot\eta$ is defined. By the Gauss–Lemma decomposition (2) is orthogonal. Hence

$$\|\dot\gamma(t)\|_{g_{\gamma(t)}} \geq \|d_{\gamma(t)} \exp_p(\dot r(t)\eta(t))\|$$
$$= |\dot r(t)| \|d_{\gamma(t)} \exp_p(\eta(t))\|$$
$$= |\dot r(t)|$$

and therefore

$$\ell(\gamma) \geq \int_0^T |\dot r(t)| dt \geq \Big| \int_0^T \dot r(t) \Big| = r(T) - r(0) = r.$$

Equality holds iff $\eta(t)$ is piecewise constant and $\dot r \geq 0$. In particular, $\dot r = 1$ wherever η is constant and hence η is constant everywhere, since γ is continuous. $\square$.

Now we are ready to derive the extremal property of geodesics.

Theorem 106: Geodesics are locally minimizing: Let (M, g) be a Riemannian manifold, $\gamma : (a, b) \to M$ be a geodesic. Then for all $t_0 \in (a, b)$ there exists an $\varepsilon > 0$ such that for all $t_1 \in (a, b)$ with $|t_1 - t_0| < \varepsilon$

$$\ell(\gamma|_{|}t_0, t_1]) = d_g(\gamma(t_0), \gamma(t_1)).$$

More precisely, for $\delta : [s_0, s_1] \to M$ piecewise differentiable and continuous with $\delta(s_0) = \gamma(t_0), \delta(s_1) = \gamma(t_1)$ we have

$$\ell(\delta) \geq \ell(\gamma|_{[t_0,t_1]})$$

with equality iff there exists a piecewise differentiable reparametrization $\tau : [s_0, s_1] \to [t_0, t_1]$, which is in particular weakly monotone, such that $\delta = \gamma \circ \tau$.

Proof: For fixed t_0 choose $\varepsilon > 0$ such that $\exp_{\gamma(t_0)}|_{B(\varepsilon)}$ is a diffeomorphism onto its image. There exists $s' \in (s_0, s_1]$ such that $\delta(s') \in S(p, \varepsilon)$ and $\delta([s_0, s']) \subset \overline{B(p, \varepsilon)}$. Then by the previous lemma

$$\ell(\delta) \geq \ell(\delta|_{[s_0,s']} \geq \varepsilon = \ell(\gamma|_{[t_0,t_1]}$$

where $t_1 = t_0 \pm \varepsilon$. Equality holds iff $\delta|_{[s',s_1]} \equiv \gamma(t_1)$ and $\delta|_{[s_0,s']}$ is a reparametrization of $\gamma|_{[t_0,t_1]}$ and the claim follows. □.

Corollary: We have for sufficiently small balls in the metric space (M, d_g)

$$\exp_p(B_{T_pM,g_p}(r)) = B_{(M,d_g)}(p, r)$$

for $r < r_p$.
In particular, the topology of the metric space (M, d_g) coincides with the original topology of the manifold M.

Proof: (1) We show that for each $p \in M$ a sufficiently small ball $B_{(M,d_g)}(p, r)$ is open in the original topology. This follows if $r < r_p$ from the identity.
(2) Let $U \subset M$ be open. Then $(U, g|_U)$ is a Riemannian manifold. Let $p \in U$. For $p \in U$ exists $\varepsilon > 0$ such that $\exp_p|_{B_{T_pM,g_p}(\varepsilon)} : B_{T_pM,g_p}(\varepsilon) \to U$ is a diffeomeorhism onto its image in, in particular, $B_{(M,d_g)}(p, \varepsilon) \subset U$. Therefore, every $p \in U$ is an interior poiint of U w.r.t. the metric d_g and hence U is open in (M, d_g). □

Proposition 107: (M, g) is a connected Riemannian manifold, $p, q \in M$. Let $\gamma : [a, b] \to M$, continuous, piecewise differentiable, $\|\dot{\gamma}(t)\| = 1$, wherever defined, $\gamma(a) = p, \gamma(b) = q$. Let γ be locally minimal. Then γ is differentiable (hence $\|\dot\gamma(t)\| = 1$ everywhere) and a geodesic. In particular, the conclusion holds if $\ell(\gamma) = d(p,q)$, i.e. γ is minimal along all such curves connecting p and q.

Proof: (1) Pick any point $t \in [a,b]$ and $\varepsilon > 0$ such that $\exp|_{\gamma(t)}|_{B(0,\varepsilon)}$ is a diffeomeorphism onto its image. By the Gauss–Lemma we conclude, that $\gamma|_{[t-\varepsilon,t]}$ and $\gamma|_{[t,t+\varepsilon]}$ are geodesics. If γ is differentiable at t, it follows, that $\gamma_{[t-\varepsilon,t+\varepsilon]}$ is a geodesic.
(2) In the situation above let us assume that γ is not differentiable at t, i.e.

$$v_+ := \lim_{s\downarrow t} \dot\gamma(s) \neq \lim_{s\uparrow t} \dot\gamma(s) =: v_-.$$

Then in $B(0,\varepsilon) \subset T_{\gamma(t)}M$ w.r.t. the euclidean norm of $g_{\gamma(t)}$ we have $\|v_+ - v_-\| =: c < 2$. On the other hand, using the result on normal coordinates we find $\|g_{\exp_p(v)} - g_p\| \leq c'\|v\|^2$ w.r.t. the same norm. Let $\varepsilon > \delta > 0$ be suficiently small and define $\tilde\gamma : [a,b] \to M$ via

$$\tilde\gamma(s) = \begin{cases} \gamma(t) & \text{if } s \notin [t-\delta, t+\delta] \\ \exp_p((s-t-\delta)/2v_- + (s-t+\delta)/2v_+ & \text{if } s \in [t-\delta, t+\delta]. \end{cases}$$

This is a continuous, piecewise differentiable curve connecting p and q. The newly inserted piece is regular (hence can be reparametrized to fulfil the requirement on the velocity) and has length

$$\ell_0 \leq c\delta\sqrt{1 + c'\delta^2}.$$

and therefore

$$\ell(\tilde\gamma) = \ell(\gamma) - \ell(\gamma|_{[t-\delta,t]}) - \ell(\gamma|_{[t,t+\delta]}) + \ell_0 = \ell(\gamma) - 2\delta + \ell_0 \leq \ell(\gamma) - \delta(2 - c\sqrt{1 + c'\delta^2}).$$

Since $c < 0$ after choosing δ sufficiently small we find

$$\ell(\tilde\gamma) < \ell(\gamma),$$

i.e. $\tilde\gamma$ shortcuts γ. Notice that the inserted piece stays in the δ–ball of $\gamma(t)$ hence contradicting local minimality property of γ. □

Theorem 108 [Hopf-Rinow]: Let (M,g) be a connected Riemannian manifold.
(a) The following conditions are equivalent:

(1) (M,g) is geodesically complete: For any $p \in M, X \in T_pM$ there exists a geodesic $\gamma : \mathcal{R} \to M$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = X$, i.e. the differential equation admits global solution on all of $\mathcal{R}$.
(2) (M, d_g) is a complete metric space.
(3) Each closed, bounded subset of (M, d_g) is compact.
(4) There exists a $p \in M$ such that $\exp_p : T_pM \to M$ is defined on the whole tangent space.

Examples: (1) Any compact metric space is complete. Hence (M,g) is complete for any compact manifold M since it is also compact w.r.t. the metric d_g.
(2) $(\mathcal{R}^n, g_{euclid.})$ and $(\mathcal{H}, g_{\mathcal{H}})$ are geodesically complete (obvious for the first, needs to be checkes for the second). Hence they are complete as metric spaces, which you knew already for the first, of course.
(3) Any metric space can be completed. Let $(\overline{M}, \overline{d_g})$ denote the completion of (M, d_g). However, (M,g) not necessarily extends to a differentiable Riemannian manifold $(\overline{M}, \overline{g})$, which can be easily seen for cones. Even worse, let N is a manifold not diffeomeorphic to a sphere, h a Riemannian metric on N. Consider the Riemannian metric g on $N \times (0,\infty)$ given by $g = (r^2 h) \oplus dr^2$ w.r.t. the splitting $T(N \times (0,\infty)) = TN \oplus \mathcal{R}$ and for parameter $r \in (0,\infty)$ and $dr^2(s,t) = st$ on the real line. Then it is not hard to see that topologically $\overline{M} = N \times [0,\infty)/N \times \{0\}$, i.e. all points of $N \times \{0\}$ are identified. This is the cone over N which is a manifold only if N is diffeomeorphic to a sphere of dimension $\dim N$.

We present the crucial idea the proof of Theorem 108 in the following

Lemma 109: For $p \in M$ let $\exp_p : T_p \to M$ be defined on all of T_pM. Recall that M was assumed to be connected. Then for any $q \in M$ there exists a minimizing geodesic $\gamma : [0, d(p,q)] \to M$ connecting p and q.

Proof: Fix $\varepsilon > 0$ such that $\exp_p |_{B_{T_pM}(0,\varepsilon)}$ is a diffeomeorphism onto its image $B_M(p,\varepsilon)$. For $q \in B_M(p,\varepsilon)$ the claim follows from the Gauss-Lemma.

Let $q \notin B_M(p,\varepsilon)$. Fix $r < \varepsilon$. Since the euclidean sphere is compact and $\exp_p$ continuous the image $S_r(p) := \exp_p(S_{T_pM}(0,r))$ is compact. Moreover, $d_g(q,.) : M \to [0,\infty)$ is continuous, hence it attains its minimum on campact subsets. Let $m \in S_r(p)$ be such that

$$d(q,m) = \min\{d(q,x) | x \in S_r(p)$$

Then $d(p,m) + d(m,q) = d(p,q)$. Indeed, any curve $\gamma : [0,T] \to M$ connecting p and q must pass through $S_r(p)$, i.e. there exists a $t_0 \in [0, T]$ such that $\gamma(t_0) \in S_r(p)$ and $\gamma(t) \in B_M(p, r)$ for $t < t_0$. Then by the Gauss–Lemma

$$\ell(\gamma) = \ell(\gamma|_{[0,t_0]}) + \ell(\gamma|_{[t_0,T]} \geq r + d(\gamma(t_0), q) = d(p, m) + d(\gamma((t_0), q)) \geq d(p, m) + d(m, q).$$

Let $v \in T_pM$ be the unique tangent vector, such that $\|v\| = 1$ and $\exp_p(v) = m$. Let $\gamma : \mathcal{R} \to M$ be the geodesic given by $\gamma(t) = \exp_p(tv)$.

Claim: $\gamma(d(p, q)) = q$!

Define the subset $I \subset [0, d(p, q)]$ via

$$I := \{t \in [0, d(p, q)] \forall s \leq t :| \; s + d(\gamma(s), q) = d(p, q)\}.$$

The claim follows from $d(p, q) \in I$.

By the discussion above $[0, \varepsilon) \subset I$.

By continuity of the metric $d(.,.)$ we conclude I is a closed subset. Hence it remains to show, that I is also open. Let $t_0 > 0$ and assume that $t_0 < d(p, q)$. Now let $0 < \delta < t_0$ such that $t_0 + \delta < d(p, q)$. We will show that $t_0 + \delta \in I$ proving $[0, t_0 + \delta] \subset I$ from which the openess of I follows.

As above let $n \in S(\gamma(t_0), \delta)$ be such that

$$d(n, q) = \min\{d(x, q) \mid x \in S_\delta(\gamma(t_0))\}.$$

Let $\gamma_0 : [0, \delta] \to B_M(\gamma(t_0), \delta)$ be the unique geodesic connecting $\gamma(t_0)$ and n (see above). Analogously, we have

$$\delta + d(n, q) = d(\gamma(t_0), q).$$

Now we have

$$t_0 + \delta + d(n, q) = t_0 + d(\gamma(t_0), q) = d(p, q)$$

since $t_0 \in I$ by assumption. Therefore

$$t_0 + \delta + d(n, q) = d(p, q) \leq d(p, n) + d(n, q)$$

and hence

$$t_0 + \delta \leq d(p, n) \leq d(p, \gamma(t_0) + d(\gamma(t_0, n) = t_0 + \delta$$

and hence

$$t_0 + \delta = d(p, n).$$

We define $\tilde{\gamma} : [0, t_0 + \delta]$ by

$$\tilde{\gamma}(t) := \begin{cases} \gamma(t) & \text{for } t \in [0, t_0] \\ \gamma_0(t - t_0) & \text{for } t \in [t_0, t_0 + \delta]. \end{cases}$$

which is continuous, piecewise differentiable with $\|\dot{\gamma}(t)\| = 1$ whereever it is defined. We have

$$\ell(\tilde{\gamma}) = t_0 + \delta = d(p, n),$$

hence minimal w.r.t. all such curves connecting p and $\gamma(t_0)$. By Proposition 107, $\tilde{\gamma}$ is a geodesic connecting the two points and since $\gamma|_{[0,t_0]} = \tilde{\gamma}|_{[0,t_0]}$ by uniqueness of solutions of ODE we conclude $\tilde{\gamma} = \gamma_{[0,t_0+\delta]}$ and, in particular, $\gamma(t_0 + \delta) = n$.

Now

$$d(p, q) = (t_0 + \delta) + d(n, q) = \ell(\tilde{\gamma}) + d(n, q) = d(p, \gamma(t_0 + \delta)) + d(\gamma(t_0 + \delta), q)$$

and hence $t_0 + \delta \in I$. □

Proof of Theorem 108:(b) follows from (a) (4) and Lemma 109.

(a) (4)⇒(3) Let $p \in M$ such that $\exp_p : T_p \to M$ is defined on all of T_pM. Let $A \subset M$ be closed and bounded w.r.t. metric d_g. A bounded means $d(p, A) = R < \infty$. Then by Lemma 108 $A \subset \exp_p(\overline{B_{T_pM}(0, R)})$. But $\overline{B_{T_pM}(0, R)} \subset T_pM$ is compact and since $\exp_p$ is continuous $\exp_p(\overline{B_{T_pM}(0, R)}) \subset M$ is compact. Since A is also assumed to be a closed subset it follows that A is compact.

(3)⇒(2) Let $\{p_n\}_n$ be a Cauchy sequence of the metric space (M, d_g). Then $A := \{p_n \mid n \in \mathbb{N}\}$ is bounded and therefore its closure $\overline{A}$ is bounded (and closed) and hence compact. In particular, a subsequence of $\{p_n\}_n$ converges in $\overline{A} \subset M$. Since $\{p_n\}_n$ was assumed to be a Cauchy sequence it follows that $\{p_n\}_n$ converges to the same limit. Hence (M, d_g) is complete.

(1)⇒(4) (1) is statement (4) for all $p \in M$.

(2)⇒(1) Let $\gamma : (a, b) \to M$ be a geodesic, $\|\dot{\gamma}\| \equiv 1$. $b < \infty$. We claim that γ can be extended as a geodesic beyond b. The argument will be similar if $a > -\infty$.

Let $t_n \in (a,b)$, such that $t_n \to b$. Then

$$d(\gamma(t_n), \gamma(t_m)) = \ell(\gamma|_{[t_n,t_m]}) = |t_n - t_m|.$$

Hence $\{\gamma(t_n)\}_n$ is a Cauchy sequence in (M, d_g) and by condition (2) there is a limit $q \in M$, i.e. $\lim_n \gamma(t_n) = q$. Let $\{t_n'\}_n$ be another such sequence. Then

$$d(\gamma(t_n'), \gamma(t_n)) = |t_n' - t_n| \to 0$$

and hence $\tilde{\gamma} : (a,b] \to M$ defined by

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t) & \text{if } t \in (a,b) \\ q & \text{if } t = b. \end{cases}$$

is continuous.

Let $\varepsilon > 0$ such that $\exp_q|_{B_{T_qM}(0,\varepsilon)}$ is a diffeomorphism onto its image $B_M(q,\varepsilon)$. By its continuity there is a $s_0 \in (a,b)$ such that

$$\tilde{\gamma}([s_0, b]) \subset B_M(q, \varepsilon).$$

Note that its length $\ell(\tilde{\gamma}) = b - s_0 = d(\gamma(s_0), q)$. Let $v \in T_qM$, $\|v_q\| = 1$ be such that $\exp_q((s_0 - b)v) = \gamma(s_0)$. Then by Theorem 107 $\tilde{\gamma}(t) = \exp_q((t-b)v)$ for $t \in [s_0, b]$ and hence $\bar{\gamma} : (a, b+\varepsilon) \to M$ with

$$\bar{\gamma}(t) := \begin{cases} \gamma(t) & \text{for } t \in (a,b) \\ \exp_q((t-b)v) & \text{for } t \in [b, b+\varepsilon) \end{cases}$$

defines a differentiable curve which is a geodesic on each part and hence a geodesic. □