Partition of Unity

Differential Geometry II
Summer 2020

Lemma 34: Let M be a smooth manifold with boundary. Let $\{U_\iota\}_{\iota\in I}$ be an open covering of M. There exist a countable family $\{\lambda_k\}_{k\in\mathbb{N}}$ of non-negative smooth functions with compact support, such that

(i) **refinement condition:** For any $k \in \mathbb{N}$ there is a $\iota \in I$ such that $\operatorname{supp}(\lambda_k) \subset U\iota$
(ii) **locally finiteness:** for each $p \in M$ there is an open subset $U \subset M$, $p \in U$ such that

$$\sharp\{k \in \mathbb{N} \mid \operatorname{supp}(\lambda_k) \cap U \neq \emptyset\} < \infty.$$

(iii) **partition of unity:**

$$\sum_{k=1}^{\infty} \lambda_k \equiv 1.$$

Proof: Since we defined manifolds slightly differently than in many "classical textbooks" we will structure the proof in such a way that it is useful in these other contexts. Most arguments are taken from Helga Baum's script with deliberatly editing if deemed appropriate.

Step 1: Let M be a topological space, which admits a countable basis $(B_k)_{k\in\mathbb{N}}$, i.e. for every open subset $U \subset M$ and $p \in U$ there is a $k \in \mathbb{N}$ such that $p \in B_k \subset U$. M is said to satisfy the **second countability axiom**. Moreover, assume that M is **locally compact**, i.e. for each $p \in M$ there is a compact neighborhood. Then for every open covering $\{W_\iota\}_{\iota\in I}$ there is a **locally finite** refinement consisting of a subfamily, $K \subset \mathbb{N}$, $\{B_\kappa\}_{\kappa\in K}$ such that the closures are compact: For each $\kappa \in K$ there is a $\iota \in I$ such that $B_\kappa \subset W_\iota$ and each $p \in M$ admits a neighbourhood N that $\{\kappa \in K | B_\kappa \cap N \neq \emptyset\}$ is finite.

Remark: Spaces admitting refinements for any open covering (without the restriction on the choice of the new open subsets as being from a given family and having compact closures) are called **paracompact**.

Proof of Step 1: For $p \in M$ we fix a neighborhood $W(p)$ with compact closure.

$$\mathcal{B} := \{B_k | \exists p \in M : B_k \subset W(p)\}$$

is a smaller countable base of M; Indeed, let $U \subset M$ be any open subset. Then

$$U = \bigcup_{x\in M} W(x) \cap U :$$

and therfore

$$U = \bigcup_{B\in\mathcal{B}, B\subset U} B.$$

In particular, for all $B \in \mathcal{B}$ the closure is contained in the closure of $W(x)$ and hence compact.

Now we define

$$\mathcal{B}' := \{B \in \mathcal{B} | \exists \iota \in I : B \subset W_\iota\} =: \{B_k\}_{k\in\mathbb{N}}.$$

Since $\{W_\iota\}$ is an open covering, this is once again a countable base of open subsets with compact closure. Now we define a nested sequence of open subsets with compact closure. $A_1 := B_1$. Let A_k

be already defined. Since its closure is compact there exist an $m_k \in \mathbb{N}$ such that

$$\overline{A_k} \subset \bigcup_{j=1}^{m_k} B_j$$

Define

$$A_{k+1} := A_k \cup \bigcup_{j=1}^{m_k} B_j.$$

Notice

$$\bigcup_{k=1}^{\infty} A_k = M,$$

and

$$\overline{A_{k+1}} \setminus A_k \subset A_{k+2} \setminus \overline{A_{k-1}}$$

are compact subsets of open subsets for all $k \in \mathbb{N}_0$, where $A_0 := A_{-1} := \emptyset$. Hence

$$\overline{A_{k+1}} \setminus A_k = \bigcup B \in \mathcal{B}', B \subset A_{k+2} \setminus \overline{A_{k-1}} B$$

ande there is a finite set $\{B_j^k\}_{j=1}^{N_k}$ of elements of $\mathcal{B}'$ contained in $A_{k+2} \setminus \overline{A_{k-1}}$ covering $\overline{A_{k+1}} \setminus A_k$.

Now, $\{B_j^k | k \in \mathbb{N}, j = 1, ..., N_k\} \subset \mathcal{B}'$ is an open covering refining $\{U_\iota\}_{\iota \in I}$, of open subsets with compact closure.

It is locally finite: Let $p \in M$. There is a $k \in \mathbb{N}_0$ such that $p \in A_{k+1} \setminus \overline{A_k}$ Since for any $\ell \geq k+2$ or $\ell \leq k-2$

$$A_{k+1} \setminus \overline{A_k} \cap A_{\ell+2} \setminus \overline{A_{\ell-1}} = \emptyset$$

we have that

$$B_j^\ell \cap A_{k+1} \setminus \overline{A_k} \neq \emptyset$$

only if $\ell = k-1, k, k+1$ which consists of only finitely many choices, and the claim follows.

Step 2: Let M be a **separabel** manifold, i.e. there is a countable, dense subset $D = \{p_k | k \in \mathbb{N}\} \subset M$. Then M satisfies the assumptions of Step 1 on its topology.

Proof of Step 2: Let $\mathcal{A} := \{(U_\nu, \varphi_\nu, V_\nu)\}_{\nu \in N}$ be a differentiable atlas of M. For $p \in M$ we define

$$R_p := \sup\{r > 0 | B(p, r) \subset U_\nu \text{ for } \nu \in N\} \in (0, \infty]$$

making use of a metric fixed on M. R_p depends continuously on p. Define

$$r_p := \min(\frac{R_p}{2}, 1).$$

Then

$$\bigcup_{k \in \mathbb{N}} B(p_k, r_{p_k}) = M.$$

Indeed: If $d(p, p_k) < \epsilon$ then $|r_{p_k} - r_p| < \frac{\epsilon}{2}$ and thus $r_{p_k} > r_p - \frac{\epsilon}{2}$. So, if $\epsilon < \frac{2}{3} r_p$ then $p \in B(p_k, r_{p_k})$ and the claim follows since D is dense. To each $n \in \mathbb{N}$ we assign a chart $(U_k, \varphi_k, V_k) \in \mathcal{A}$ such that $B(p_k, r_{p_k}) \subset U_k$. Then

$$\{(U_k, \varphi_k, V_k)\}_{k \in \mathbb{N}}$$

is a countable differentiable atlas of M. In particular,

$$\{\varphi_k(B(x, r)) | k \in \mathbb{N}, x \in V_k \cap \mathbb{Q}^n, r \in \mathbb{Q}_+, B(x, r) \subset V_k\}$$

is a countable base of the topology of M. Each point admits a neighbourhood homeomorphic to a closed disk and is hence compact.

From Step 1 follows that for each open covering $\{W_\iota\}_{\iota\in I}$ there is a locally finite subset

$$\{B_k := \varphi_k(B(x_k, r_k))|k \in \mathbb{N}\}$$

of the base (hence countable) such that for each $k \in \mathbb{N}$ there is a $\iota_k \in I$ such that $B_k \subset U_{\iota_k}$.

Remark: The elements of the basis $\{B_k\}_{k\in\mathbb{N}}$ we constructed have compact closure by construction. We proved a statement more general than needed in Step 1 to be aligned with the literature (see Helga Baums Script, Satz 1.27). An easy exercise left is to shorten the whole proof a bit by taking advantage of this fact.

Step 3: There are smooth functions $\tilde{\lambda}_k : M \to [0,1]$ with support $\mathrm{supp}\tilde{\lambda}_k = B_k$ and $\tilde{\lambda}_k|_{B_k} > 0$.

Proof of Step 3: $B_k = \varphi_k(B(x_k, r_k))$ for the chart (U_k, φ_k, V_k) assigned to B_k. We start with the well–known fact, that via

$$f(x) := \begin{cases} e^{-\frac{1}{x}} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

we define a smooth function $f : \mathbb{R} \to [0,1]$. Then

$$\tilde{\lambda}_k(p) = f(r_k^2 - (\varphi_k^{-1}(p) - x_k)^2).$$

on B_k and zero outside satisfies the required properties.

Finally,

$$\lambda_k = \frac{\tilde{\lambda}_k}{\sum_{k=1}^{\infty} \tilde{\lambda}_k}.$$

By construction all three properties of the Lemma are satisfied.