Problem Set 6

Differential Geometry II
Summer 2020

Problem 1 [Hermitian Connections]
(a) Show that the curvature of a Hermitian connection of a Hermitian vector bundle is skew-symmetric w.r.t. the Hermitian form.
(b) Show that the connection 1–form and the curvature w.r.t. a Hermitian trivialization satisfy

$$A_k^\ell = -\overline{A_\ell^k} \quad \text{and } F_k^\ell = -\overline{F_\ell^k}.$$

Problem 2 [Tautological Line Bundle]
(a) Show that

$$H := \{([z_1, z_2], (\lambda z_1, \lambda z_2)) | [z_1, z_2] \in \mathbb{C}P^1, \lambda \in \mathbb{C}\}$$

is a smooth vector bundle. Describe a trivialization on

$$U_k := \{[z_1, z_2] \in \mathbb{C}P^1 | z_k \neq 0\}$$

for $k = 1, 2$ and the corresponding transition function. Can you explain, why this bundle must be non-trivial?
(b) A section $\sigma : U \to H|_U \subset U \times \mathbb{C}^2$ can be considered as two complex funtions. We define a connection ∇ on the bundle by

$$\nabla\sigma := \mathrm{proj}_H^\perp(d\sigma)$$

where $\mathrm{proj}_H^\perp$ denotes the orthogonal projection w.r.t. the standard scalar product. Show that this is a complex connection, i.e. satifies Leibniz' Rule even for complex valued smooth functions. Express it in the trivializations found in (a). Hint: Make use of the Hermitian product on $\mathbb{C}^2$.
(c) Compute the curvature of ∇. Explain that it is a 2–form on $\mathbb{C}P^1$ (with purely imaginary values).
(d) Consider the Hopf bundle and the representation $\rho : S^1 \to Aut(\mathbb{C}) = \mathbb{C} \setminus \{0\}$ given by $\rho(g) = g$. Show that

$$S^3 \times_\rho \mathbb{C} \cong H.$$

Problem 3 [Lie Groups]
(a) Show that $O(n)$ and $SO(n)$ are Lie groups and their Lie algebras are given by

$$\underline{o}(n) = \underline{so}(n) := \{A \in M(n; \mathbb{R}) \mid A^T = -A\}.$$

(b) Show that the Lie bracket on $\underline{o}(n)$ is determined by matrix multiplication: for $X, Y \in \underline{o}(n)$

$$[X, Y] = XY - YX.$$

(c) Show that via

$$\langle A, B\rangle := -Trace(AB)$$

and the by left action given by $L_g(h) = gh$ one defines a Riemannian metric on G. L_g is an isometry for all g by definition. Show that this is also true for the right action R_g defined by $R_g(h) = hg$.

Problem 4 [Riemann Surfaces]
(a) Show that the coordinate changes of a complex atlas of a Riemann surface are bi-holomorphic functions, i.e. the transition function together with its inverse satisfy Cauchy–Riemann equations.
(b) Show that the Nijenhuis–tensor on a Riemann surface always vanishes (without referring to the existence of a complex atlas).

Problems 5 and 6 will be only discussed if there are attempts, ideas of concrete questions.

Problem 5 [Kähler Manifolds]
Let (M, J, g) be a Hermitian manifold. Show that $N_J \equiv 0$ and $d\omega = 0$ for the Kähler form ω implies $\nabla J \equiv 0$. For this express $2g((\nabla_X J)Y, Z)$ in terms of $d\omega$ and $g(., N_J(., .))$ evaluated on X, Y, Z and possibly JX, JY, JZ.

Problem 6[Complex Manifolds]
(a) Show that the Nijenhuis-tensor is a tensor, i.e. defines a bilinear map

$$N_{J,p} : T_pM \times T_pM \to T_pM.$$

(b) Let M be a manifold with a complex atlas $\{U_\iota, \varphi_\iota, V_\iota\}_\iota$, i.e. the coordinate changes

$$\varphi_\kappa^{-1} \circ \varphi_\iota : \varphi_{-1} ota^{-1}(U_\iota \cap U_\kappa) \subset \mathbb{C}^n \to \varphi_\kappa^{-1}(U_\iota \cap U_\kappa) \subset \mathbb{C}^n$$

are holomorphic in each component. Define $\{J_p\}_{p\in M}$ by

$$d_{\varphi_\iota^{-1}(p)}\varphi_\iota \mathrm{i} d_p(\varphi_\iota^{-1}) : T_pM \to T_pM$$

when $p \in U_\iota$ where i between the differentials denotes multiplication by the imeginary unit. Show that $J^2 = -\mathrm{id}$. Prove the indepence of the definition from the particular chart chosen for which $p \in U_\iota$. Such manifolds are called complex manifolds.
(c) Show that $N_J \equiv 0$ for a complex manifold vanishes.