Differential Geometry II

Curvature

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The Space of Connections

$E \to M$ vector bundle

$E_p = \pi^{-1}(p)$

The set of all connections

$$\mathcal{C}(E) = \{\nabla | \nabla \text{ connection of } E\}$$

is an affine space over

$$\begin{aligned} \alpha \in \Omega^1(M; End(E)) &= \Gamma(T^*M \otimes End(E)) \\ &= \{\sigma : M \to T^*M \otimes \underline{E^* \otimes E} \mid \sigma \text{ smooth section}\}. \end{aligned}$$

if $X \in T_pM$ $\quad \alpha_p(X) \in End(E_p)$

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Indeed: Let ∇^0, ∇ be connections on the vector bundle $E \xrightarrow{\pi} M$.
Consider $\alpha := \nabla - \nabla^0 : \Gamma(E) \to \Omega^1(E)$.

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$\sigma : M \to E$ section, smooth
$f \in C^\infty(M)$

$$\alpha(f\sigma) = \nabla(f\sigma) - \nabla^0(f\sigma) = f(\nabla\sigma - \nabla^0\sigma) + (df)\sigma - (df)\sigma = f\alpha(\sigma)$$

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It follows: for $\sigma \in \Gamma(U, E|_U)$ with $\sigma(p) = 0$ we have $\alpha(\sigma)(p) = 0$.

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For $v \in E_p$ let $\sigma \in \Gamma(U, E|_U)$ sich that $\sigma(p) = v$ and define

$$\alpha_p(v) := \alpha(\sigma)(p).$$

Pull-Backs

Denote by (E, ∇) a vector bundle of rank k over a manifold M equipped with a connection ∇. Let $g : P \to M$ be a smooth map between manifolds (with boundary).

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Definition 53: (1) The **pull back**, g^*E, of the bundle E is the vector bundle

$$g^*E = \coprod_{p \in P} E_{g(p)} \xrightarrow{\pi} P$$

where a trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ of E over $U \subset M$ induces a trivialization $\Phi_g : g^{-1}(\pi^{-1}(U)) \to g^{-1}(U) \times \mathbb{R}^k$ via

$$\Phi_g(e) = (p, \mathrm{pr}_{\mathbb{R}^k}\Phi(e)) \qquad \subset P \text{ open}$$

for $e \in (g^*E)_p$. $= E_{g(p)}$ $\quad$ ($p_1 \neq p_2$ & $g(p_1) = g(p_2)$) $(g^*E)_{p_1} \neq (g^*E)_{p_2}$

$\{U_i\}_i$ covering of M $\Rightarrow$ $\{g^{-1}(U_i)\}_i$ covering of P

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for $e \in (g^*E)_p$.

(2) The **pull back**, ∇^g, of the connection ∇ is given w.r.t. the trivialization by the connection 1–form

$$A_\Phi^g := g^* A_\Phi.$$

... pull-back of 1-forms

Parallel Transport

∇^g is well–defined, i.e. independent of the local trivialization Φ of E.

If for two trivializations Φ, Ψ

$$\Psi \circ \Phi^{-1}(x, v) = (x, \varphi(x) v)$$

$$\varphi : U \cap V \to Gl(k, \mathbb{R}) \quad \text{transition function}$$

$$\Leftrightarrow \quad \varphi \circ g : g^{-1}(U) \cap g^{-1}(V) \to Gl(k, \mathbb{R})$$

transition functions for triv. of g^*E

$$A^g_\Psi = (\varphi \circ g)^{-1} A^g_\Phi (\varphi \circ g) + (\varphi \circ g)^{-1} d(\varphi \circ g)$$

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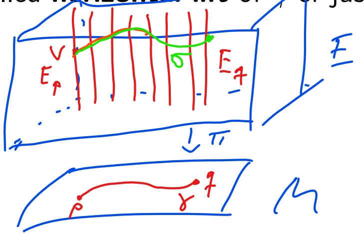
σ is called **horizontal lift** of γ or just **horizontal curve**.



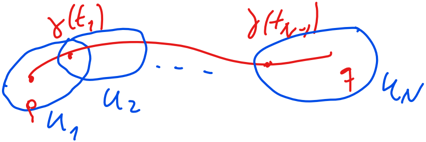
Parallel Transport

Let $(U_i)_{i\in I}$ open covering of M.

$\Rightarrow$ Then $(\gamma^{-1}(U_i))_{i\in I}$ open covering of $[a,b]$

$[a,b]$ compact $\Rightarrow$ $\exists$ finite open subcovering $\{\gamma^{-1}(U_k)\}_{k=1}^{N}$.

$\exists$ $a=t_0<t_1<\dots=t_N=b$ s.t. $[t_k,t_{k+1}]\subset\gamma^{-1}(U_k)$



On $[t_k,t_{k+1}]$ $\quad s_k := \Phi_k\circ\sigma|_{[t_k,t_{k+1}]} : [t_k,t_{k+1}]\to\mathbb{R}^k$

$\Phi_k\circ\nabla^{\gamma}_{\frac{\partial}{\partial t}}\sigma : [t_k,t_{k+1}]\to\mathbb{R}^k \qquad \Phi_k\circ\nabla^{\gamma}_{\frac{\partial}{\partial t}}\sigma = \frac{ds_k(t)}{dt} + A^{\gamma}_{\Phi_k}(\dot\gamma(t))\,s_k(t)=0$

is a linear ODE: global solutions for any initial value.

Legend: First solve it on $[t_0,t_1]$ with $s_0(t_0)=\Phi_0(v)$ ..., on $[t_k,t_{k+1}]$, $s_k(t_k)=s_{k-1}(t_k)$

Horizontal Spaces

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(i) A covariant derivative ∇ on sections of E

(ii) A **horizontal splitting** $T_eE = T_e^hE \oplus E_{\pi(e)}$ which depends smoothly on e and satisfies for $\mu_\lambda : E \to E$, $\mu_\lambda(e) = \lambda e$

$$d_e\mu_\lambda(T_e^hE) = T_{\lambda e}^hE.$$

Proof: (i)$\Rightarrow$ (ii): Given a covariant derivatve, we define for $e \in E_p$

$$T_e^hE := \{\dot\sigma(0) \mid \sigma : I \to E, 0 \in I, \text{ horizontal}\}$$
$$= \{(d_e\Phi)^{-1}(X, v) \mid X \in T_pU, v \in \mathbb{R}^k, A_{\Phi,p}(X)\Phi(e) + v = 0\} \subset T_eE$$

for a trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and the connection 1–form A_Φ of ∇ w.r.t. Φ.

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Then $d_{(p,x)} m_\lambda(X, v) = (X, \lambda v)$ and

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and the claim follows.

Remark: The horizontal splitting also satisfies an addition property. This follows from direct calculation or from the scaling invariance just proved via an argument along Chris Wendl's script, Chapter 3.3. : $P_e : T_e E \to E_e$ projection map induced by splitting.

P_e depends smoothly on e & $P_e|E_e = id_{E_e}$

& " $P_{\lambda e} \circ d_e \mu_\lambda = \mu_\lambda \circ P_e$ "

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(ii)$\Rightarrow$(i): ∇ defined via

$$(\nabla\sigma)_p := \mathrm{pr}_{E_p} d_p\sigma$$

$d_p\sigma : T_pM \to T_{\sigma(p)}$

on sections $\sigma : U \to E|_U$ where pr_{E_p} is the projection with respect to the splitting is a covariant derivative whose horizontal splitting is the given one.

Curvature

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$\alpha \in$

$X_1, \dots X_k$ vect. fields on $U \subset M$ open.

$\Rightarrow \alpha(X_1, \dots, X_k) : U \to E$ section

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$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E) = \Omega^1(M, E)$

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$$D_{k+\ell}(\alpha \wedge \sigma) = d\alpha \wedge \sigma + (-1)^k \alpha \wedge D_\ell \sigma$$

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Proposition 56: There is an element $F = F^\nabla \in \Omega^2(End(E))$ such that for any $\sigma \in \Omega^k(M,E)$

$$D(D\sigma) =: D^2\sigma = F^\nabla \wedge \sigma.$$

$\in End(E_p)$ F_p

$$(F^\nabla \wedge \sigma)(X_1, \ldots, X_{k+2}) = \frac{1}{2!\,k!} \sum_{\sigma \in S_{k+2}} F_p^\nabla(X_{\sigma(1)}, X_{\sigma(2)})\big(\sigma(X_{\sigma(3)}, \ldots, X_{\sigma(k+2)})\big)$$

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$$D^2\sigma = F^\nabla \wedge \sigma.$$

Proof: $k=0$, $\sigma \in \Gamma(E)$
$D^2\sigma(X,Y) = (*)$
$D^2(f\sigma) = f \cdot D^2\sigma \;\; \forall f \in C^\infty$
$\Rightarrow D^2_p(X,Y) : E_p \to E_p$ well-def.
$k>0$ use Leibniz-Rule (Exercise) □

In particular, we have

$$(*) \quad F^\nabla(X,Y)\sigma = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})\sigma.$$

for any $\sigma \in \Gamma(E)$ and vector fields X, Y. on M

Curvature

F^∇ is the **curvature of** ∇.

Proposition 57: (1) Let $A \in \Omega^1(U; M(n;\mathbb{R}))$ be the connection 1-form w.r.t. a trivialization. Then for the curvature we have

$$F_A = dA + A \wedge A \in \Omega^2(U; M(n;\mathbb{R}))$$

w.r.t. the trivialization. Hereby with $A = (A^i_j)$ in $\Omega^1(U))$

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(2) Let ∇^0, ∇ be two connections, $\nabla = \nabla^0 + \alpha$, for $\alpha \in \Omega^1(M; End(E))$. Then

$$F^\nabla = F^{\nabla^0} + D^0\alpha + \alpha \wedge \alpha .$$

$D^0 : \Omega^1(M, End(E)) \to \Omega^2(M, End(E))$ induced by ∇^0

∇^0 connection on E ⇝ connection on $E^* \otimes E = End(E)$

⇝ D^0 on $\Omega^k(M, End(E))$

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(3) Let $p \in M$, $e \in E_p$, X, Y be two vector fields on M in a neighbourhood of p. Let $\tilde{X}, \tilde{Y}$ be their horizontal lifts to E. Then

$\tilde{X}_e = (d\pi|T_e E)^{-1}(X_p)$

$$F^{\nabla}(X, Y)e = [\tilde{X}, \tilde{Y}]_e$$

Curvature

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Proof:

Curvature

2nd Bianchi Identity

Proposition 58: With the notation from above we have

$$DF^{\nabla} = 0.$$

Proof:

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A **metric connection** on a euclidean vector bundle (E, g) is a covariant derivative ∇ which satisfies in addition

$$d(g(\sigma, \tau)) = g(\nabla \sigma, \tau) + g(\sigma, \nabla \tau)$$

for any pair of (local) sections σ, τ of E.

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(2) The parallel transport of a metric connection defines isometries between the fibres.
(3) The curvature F of a metric connection is skew-symmetric:

$$g(F(e), f) = -g(e, F(f)).$$

Euclidean Vector Bundles

A euclidean vector bundle can be locally trivialised by isometries:

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$$

such that

$$\Phi|_{E_p} : (E_p, g_p) \longrightarrow (\mathbb{R}^k, \langle ., . \rangle)$$

is an isometry for all $p \in U$. Φ will be called **euclidean trivialization**.

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In particular, the transition functions are smooth maps

$$g : U \cap V \to O(k)$$

to the set of orthogonal matrices.

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An **oriented** vector bundle is a choice of orientations of all E_p such that the trivializations Φ can be chosen, so that

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is orientation preserving w r.t. the standard orientation of $\mathbb{R}^k$.

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For an oriented euclidean vector bundle and oriented, euclidean trivializatons, the transition functions are smooth maps

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Vice versa: A family of transition funtions

$$g_{ij} : U_i \cap U_j \to O(k) \text{ or } SO(k)$$

for an open covering $\{U_i\}_{i \in I}$ satsifying the cocycle condition defines an (oriented) euclidean vector bundle over M up to (orientation) and metric preserving ismomorphisms (short: isometries).

Euclidean Vector Bundles

The connection 1-form and the curvatire of a metric connection satisfy

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One defines for $A = \sum_i A_i dx^i$ and $B = \sum_i B_i dx^i$

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with $[X, Y] = XY - YX$.

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Complex Vector Bundles

Definition 60: A **complex vector bundle** is a (real) vector bundle $E \xrightarrow{\pi} M$ together with a smooth family $\{J_p\}_{p \in M}$ of complex structures $J_p \in End_{\mathbb{R}}(E_p)$, $J_p^2 = -\mathrm{id}_{E_p}$.

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Vice versa: A family of such transition functions satisfying the cocycle condition defines a complex vector bundle up to isomorphism.