Differential Geometry II

Euclidean, Complex and Hermitian Structures

Klaus Mohnke

May 28, 2020

Curvature

Let ∇ be a connection on the vector bundle $E \xrightarrow{\pi} M$ rk E = k

Proposition 57: (1) Let $A \in \Omega^1(U; M(n;\mathbb{R}))$ be the connection 1-form w.r.t. a trivialization. Then for the curvature we have

$$F^\nabla =: F_A = dA + A \wedge A \in \Omega^2(U; M(n;\mathbb{R}))$$

w.r.t. the trivialization. Hereby with $A = (A^i_j) \in \Omega^1(U))$

$$(A \wedge A)^i_j = \sum_{\ell=1}^{k} A^i_\ell \wedge A^\ell_j .$$

Curvature

Let ∇ be a connection on the vector bundle $E \xrightarrow{\pi} M$

Proposition 57: (1) Let $A \in \Omega^1(U; M(n;\mathbb{R}))$ be the connection 1-form w.r.t. a trivialization. Then for the curvature we have

$$F^{\nabla} =: F_A = dA + A \wedge A \in \Omega^2(U; M(n;\mathbb{R}))$$

w.r.t. the trivialization. Hereby with $A = (A^i_j) \in \Omega^1(U))$

$$(A \wedge A)^i_j = \sum_{\ell=1}^{k} A^i_\ell \wedge A^\ell_j.$$

(2) Let ∇^0, ∇ be two connections, $\nabla = \nabla^0 + \alpha$, for $\alpha \in \Omega^1(M; End(E))$. Then

$$F^{\nabla} = F^{\nabla^0} + D^0\alpha + \alpha \wedge \alpha.$$

∇ connection on $E \rightsquigarrow \nabla$ induces connection on $End(E)$:
if $\sigma \in \Gamma(E)$ section, $\phi \in \Gamma(End(E))$
$(\nabla_X \phi)(\sigma) := \nabla_X(\phi(\sigma)) - \phi(\nabla_X \sigma)$
$\rightsquigarrow$ exterior covariant derivatives $D: \Omega^k(M, End(E)) \to \Omega^{k+1}(M, End(E))$

Curvature

Let ∇ be a connection on the vector bundle $E \xrightarrow{\pi} M$

Proposition 57: (1) Let $A \in \Omega^1(U; M(n;\mathbb{R}))$ be the connection 1-form w.r.t. a trivialization. Then for the curvature we have

$$F^\nabla =: F_A = dA + A \wedge A \in \Omega^2(U; M(n;\mathbb{R}))$$

w.r.t. the trivialization. Hereby with $A = (A^i_j) \in \Omega^1(U))$

$$(A \wedge A)^i_j = \sum_{\ell=1}^{k} A^i_\ell \wedge A^\ell_j.$$

(2) Let ∇^0, ∇ be two connections, $\nabla = \nabla^0 + \alpha$, for $\alpha \in \Omega^1(M; End(E))$. Then

$$F^\nabla = F^{\nabla^0} + D^0\alpha + \alpha \wedge \alpha.$$

(3) Let $p \in M$, $e \in E_p$, X, Y be two vector fields on M in a neighbourhood of p. Let $\tilde{X}, \tilde{Y}$ be their horizontal lifts to E, $\tilde{X}_e = (d_e\pi|_{T^heE})(X_{\pi(e)})$. Then

$$F^\nabla(X, Y)e = [\tilde{X}, \tilde{Y}]_e.$$

Curvature

Note: $A \wedge A \neq 0$ in general! As opposed to $\alpha \wedge \alpha = 0$ for $\alpha \in \Omega^1(M)$

Proof of (1): In trivialization $\Phi \circ \nabla \circ \Phi^{-1}$ on $C^\infty(U, \mathbb{R}^k)$ $= d + A$

$$\rightsquigarrow \Phi \circ \mathcal{D} \circ \Phi^{-1} = d + A \wedge$$

$$\Phi \circ F^\nabla \circ \Phi^{-1} \sigma = \Phi \circ \mathcal{D}^2 \circ \Phi^{-1} \sigma = (d + A\wedge) \circ (d + A) \sigma$$

$\sigma \in \Omega^0(U, \mathbb{R}^k)$

$$= \underbrace{d^2}_{=0} + \underline{(A\wedge) \circ d\sigma + d(A\sigma)} + A \wedge A \sigma$$

$$+ A \wedge d\sigma + dA\,\sigma - A \wedge d\sigma + A \wedge A \sigma$$

$$= (dA + A \wedge A)\sigma \qquad \square$$

Curvature

Note: $A \wedge A \neq 0$ in general!

Proof of Proposition 57:

Curvature

2nd Bianchi Identity

Proposition 58: With the notation from above we have

$$DF^{\nabla} = 0.$$

D acts on $\Omega^2(M, End(E))$
exter. cov. deriv. induced by ∇

Proof:

$DF^{\nabla} \in \Omega^3(M, End(E))$, $\sigma \in \Gamma(E)$ "test section"

$DF^{\nabla}(\sigma) \in \Omega^3(M, E)$

$$" D(\underbrace{F^{\nabla}(\sigma)}_{\in \Omega^2(M,E)}) - F^{\nabla} \wedge (D\sigma) = D(D^2\sigma) - D^2(D\sigma)$$

$$= D^3\sigma - D^3\sigma = 0 \quad \square$$

Euclidean Vector Bundles

Definition 59: Let $E \stackrel{\pi}{\to} M$ be a smooth vector bundle. A euclidean structure on E is a smooth family $\{g_p\}_{p\in M}$ of scalar products on the fibres E_p.

Euclidean Vector Bundles

Definition 59: Let $E \xrightarrow{\pi} M$ be a smooth vector bundle. A euclidean structure on E is a smooth family $\{g_p\}_{p \in M}$ of scalar products on the fibres E_p.

A **metric connection** on a euclidean vector bundle (E, g) is a covariant derivative ∇ which satisfies in addition

$$d(g(\sigma, \tau)) = g(\nabla\sigma, \tau) + g(\sigma, \nabla\tau)$$

for any pair of (local) sections σ, τ of E.

Euclidean Vector Bundles

Definition 59: Let $E \stackrel{\pi}{\to} M$ be a smooth vector bundle. A euclidean structure on E is a smooth family $\{g\}_{p\in M}$ of scalar products on the fibres E_p.

A **metric connection** on a euclidean vector bundle (E, g) is a covariant derivative ∇ which satisfies in addition

$$d(g(\sigma,\tau)) = g(\nabla\sigma,\tau) + g(\sigma,\nabla\tau)$$

for any pair of (local) sections σ, τ of E.

Remark: (1) The metric condition is much harder to define in terms of the horizontal vector spaces of TE.

Euclidean Vector Bundles

Definition 59: Let $E \xrightarrow{\pi} M$ be a smooth vector bundle. A euclidean structure on E is a smooth family $\{g\}_{p \in M}$ of scalar products on the fibres E_p.

A **metric connection** on a euclidean vector bundle (E, g) is a covariant derivative ∇ which satisfies in addition

$$d(g(\sigma, \tau)) = g(\nabla\sigma, \tau) + g(\sigma, \nabla\tau)$$

for any pair of (local) sections σ, τ of E.

Remark: (1) The metric condition is much harder to define in terms of the horizontal vector spaces of TE.
(2) The parallel transport of a metric connection defines isometries between the fibres.

Euclidean Vector Bundles

Definition 59: Let $E \stackrel{\pi}{\to} M$ be a smooth vector bundle. A euclidean structure on E is a smooth family $\{g\}_{p \in M}$ of scalar products on the fibres E_p.

A **metric connection** on a euclidean vector bundle (E, g) is a covariant derivative ∇ which satisfies in addition

$$d(g(\sigma, \tau)) = g(\nabla\sigma, \tau) + g(\sigma, \nabla\tau)$$

for any pair of (local) sections σ, τ of E.

Remark: (1) The metric condition is much harder to define in terms of the horizontal vector spaces of TE.
(2) The parallel transport of a metric connection defines isometries between the fibres.
(3) The curvature F of a metric connection is skew-symmetric:

$$g_p(F_p(e), f) = -g_p(e, F_p(f)). \qquad e, f \in E_p$$

Euclidean Vector Bundles

A euclidean vector bundle can be locally trivialised by isometries:

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$$

such that

$$\Phi|_{E_p} : (E_p, g_p) \longrightarrow (\mathbb{R}^k, \langle ., . \rangle)$$

is an isometry for all $p \in U$. Φ will be called **euclidean trivialization**.

Start with trivialization $\tilde{\Phi} : \pi^{-1}(U) \to U \times \mathbb{R}^k$

$v_i := \tilde{\Phi}^{-1}(\cdot, e_i) \in \Gamma(U, E|_U)$

→ $\{\tilde{v}_1, \dots, \tilde{v}_k\}$ forms basis of E_p $\forall p \in U$.

apply Gram-Schmidt: $\{v_1, \dots, v_k\} \subset \Gamma(U, E|_U)$ smooth forms orthonormal basis. → $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ as required

Euclidean Vector Bundles

A euclidean vector bundle can be locally trivialised by isometries:

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$$

such that

$$\Phi|_{E_p} : (E_p, g_p) \longrightarrow (\mathbb{R}^k, \langle .,. \rangle)$$

is an isometry for all $p \in U$. Φ will be called **euclidean trivialization**.

In particular, the transition functions are smooth maps

$$g : U \cap V \to O(k) = \{ A \in M(k, \mathbb{R}) \mid A^T A = E_k \}$$

to the set of orthogonal matrices.

Euclidean Vector Bundles

An **oriented** vector bundle is a choice of orientations of all E_p such that the trivializations Φ can be chosen, so that

$$\Phi|_{E_p} : E_p \to \mathbb{R}^k$$

is orientation preserving w r.t. the standard orientation of $\mathbb{R}^k$.

Euclidean Vector Bundles

An **oriented** vector bundle is a choice of orientations of all E_p such that the trivializations Φ can be chosen, so that

$$\Phi|_{E_p} : E_p \to \mathbb{R}^k$$

is orientation preserving w r.t. the standard orientation of $\mathbb{R}^k$.

For an oriented euclidean vector bundle and oriented, euclidean trivializatons, the transition functions are smooth maps

$$g : U \cap V \to SO(k)$$

to the set of orthogonal matrices with determinant equal to 1.

Euclidean Vector Bundles

An **oriented** vector bundle is a choice of orientations of all E_p such that the trivializations Φ can be chosen, so that

$$\Phi|_{E_p} : E_p \to \mathbb{R}^k$$

is orientation preserving w r.t. the standard orientation of $\mathbb{R}^k$.

For an oriented euclidean vector bundle and oriented, euclidean trivializatons, the transition functions are smooth maps

$$g : U \cap V \to SO(k)$$

to the set of orthogonal matrices with determinant equal to 1.

Vice versa: A family of transition funtions

$$g_{ij} : U_i \cap U_j \to O(k) \text{ or } SO(k)$$

for an open covering $\{U_i\}_{i \in I}$ satsifying the cocycle condition defines an (oriented) euclidean vector bundle over M up to (orientation) and metric preserving ismomorphisms (short: isometries).

Euclidean Vector Bundles

The connection 1-form and the curvature of a metric connection satisfy

$$A^i_j = -A^j_i \text{ and } F^i_j = -F^j_i$$

w.r.t. a euclidean trivialization, i.e.

Euclidean Vector Bundles

The connection 1-form and the curvature of a metric connection satisfy

$$A^i_j = -A^j_i \text{ and } F^i_j = -F^j_i$$

w.r.t. a euclidean trivialization, i.e.

$$A \in \Omega^1(U; \underline{o}(n)), \quad F \in \Omega^2(U; \underline{o}(n))$$

where $\underline{o}(n) \subset M(k; \mathbb{R})$ denotes the set of skew–symmetric matrices.

$$\underline{o}(k) = \{ A \in M(k, \mathbb{R}) \mid A^T = -A \} \; (= T_{1_k} O(k))$$

Euclidean Vector Bundles

The connection 1-form and the curvature of a metric connection satisfy

$$A^i_j = -A^j_i \text{ and } F^i_j = -F^j_i$$

w.r.t. a euclidean trivialization, i.e.

$$A \in \Omega^1(U; \underline{o}(n)), \quad F \in \Omega^2(U; \underline{o}(n))$$

where $\underline{o}(n) \subset M(k; \mathbb{R})$ denotes the set of skew–symmetric matrices.

Note: For $A, B \in \Omega^1(U; \underline{o}(n))$ in general $A \wedge B \notin \underline{o}(n)$

Euclidean Vector Bundles

The connection 1-form and the curvature of a metric connection satisfy

$$A^i_j = -A^j_i \text{ and } F^i_j = -F^j_i$$

w.r.t. a euclidean trivialization, i.e.

$$A \in \Omega^1(U; \underline{o}(n)), \quad F \in \Omega^2(U; \underline{o}(n))$$

where $\underline{o}(n) \subset M(k; \mathbb{R})$ denotes the set of skew–symmetric matrices.

Note: For $A, B \in \Omega^1(U; \underline{o}(n))$ in general $A \wedge B \notin \underline{o}(n)$ $\Omega^2(U, \underline{o}(n))$
but $A \wedge A \in \Omega^2(U; \underline{o}(n))$.

Euclidean Vector Bundles

The connection 1-form and the curvature of a metric connection satisfy

$$A^i_j = -A^j_i \text{ and } F^i_j = -F^j_i$$

w.r.t. a euclidean trivialization, i.e.

$$A \in \Omega^1(U; \underline{o}(n)), \quad F \in \Omega^2(U; \underline{o}(n))$$

where $\underline{o}(n) \subset M(k; \mathbb{R})$ denotes the set of skew–symmetric matrices.

Note: For $A, B \in \Omega^1(U; \underline{o}(n))$ in general $A \wedge B \notin \underline{o}(n)$ but $A \wedge A \in \Omega^2(U; \underline{o}(n))$.

$A \wedge A = \sum_{i,j} A_i A_j \, dx^i \wedge dx^j = \sum_{i<j} (A_i A_j - A_j A_i) \, dx^i \wedge dx^j$, where $A_i A_j - A_j A_i = [A_i, A_j]$

One defines for $A = \sum_i A_i dx^i$ and $B = \sum_i B_i dx^i$

$$[A, B] = [A \wedge B] = \sum_{i,j} [A_i, B_j] dx^i \wedge dx^j$$

with $[X, Y] = XY - YX$.

Euclidean Vector Bundles

The connection 1-form and the curvature of a metric connection satisfy

$$A^i_j = -A^j_i \text{ and } F^i_j = -F^j_i$$

w.r.t. a euclidean trivialization, i.e.

$$A \in \Omega^1(U; \underline{o}(n)), \quad F \in \Omega^2(U; \underline{o}(n))$$

where $\underline{o}(n) \subset M(k; \mathbb{R})$ denotes the set of skew–symmetric matrices.

Note: For $A, B \in \Omega^1(U; \underline{o}(n))$ in general $A \wedge B \notin \underline{o}(n)$ but $A \wedge A \in \Omega^2(U; \underline{o}(n))$.

One defines for $A = \sum_i A_i dx^i$ and $B = \sum_i B_i dx^i$

$$[A, B] = [A \wedge B] = \sum_{i,j} [A_i, B_j] dx^i \wedge dx^j$$

with $[X, Y] = XY - YX$. Then $A \wedge A = \frac{1}{2}[A, A]$ and $F = dA + \frac{1}{2}[A, A]$. *very usual notation*

Complex Vector Bundles

Definition 60: A **complex vector bundle** is a (real) vector bundle $E \xrightarrow{\pi} M$ together with a smooth family $\{J_p\}_{p \in M}$ of complex structures $J_p \in End_{\mathbb{R}}(E_p)$, $J_p^2 = -\mathrm{id}_{E_p}$, i.e. each fibre E_p is a complex vector space.

$\lambda = a + ib$, $e \in E_p$ define a $\mathbb{C}$-vector space E_p

$\lambda \cdot e := ae + b\,J_p(e)$

Complex Vector Bundles

Definition 60: A **complex vector bundle** is a (real) vector bundle $E \xrightarrow{\pi} M$ together with a smooth family $\{J_p\}_{p \in M}$ of complex structures $J_p \in End_{\mathbb{R}}(E_p)$, $J_p^2 = -\mathrm{id}_{E_p}$, i.e. each fibre E_p is a complex vector space.

In particular, the real rank is even and the complex rank is defined to be

$$k := \mathrm{rk}_{\mathbb{C}} E := \frac{\mathrm{rk}_{\mathbb{R}} E}{2}.$$

Complex Vector Bundles

Definition 60: A **complex vector bundle** is a (real) vector bundle $E \xrightarrow{\pi} M$ together with a smooth family $\{J_p\}_{p\in M}$ of complex structures $J_p \in End_{\mathbb{R}}(E_p)$, $J_p^2 = -\mathrm{id}_{E_p}$, i.e. each fibre E_p is a complex vector space.

In particular, the real rank is even and the complex rank is defined to be

$$k = \mathrm{rk}_{\mathbb{C}} E = \frac{\mathrm{rk}_{\mathbb{R}} E}{2}.$$

There exist local trivializations $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ such that

$$\Phi|_{E_p} : (E_p, J_p) \to \mathbb{C}^k$$

is complex linear for all $p \in U$.

Let $\tilde{\Phi} : \pi^{-1}(U) \to U \times \mathbb{R}^{2k}$ be a trivialization of the real vector bundle.

$\leadsto \{\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_{2k}\} \subset \Gamma(U, E|_U)$

$p \in U$: $\exists\, i_1, \dots, i_k$ s.t. $\{\tilde{v}_{i_1}(p), \dots, \tilde{v}_{i_k}(p)\}$ $\mathbb{C}$-basis of E_p

$\leadsto \exists$ open nbhd. $V \subset U$ of p s.t. $\{\tilde{v}_{i_1}|_V, \dots, \tilde{v}_{i_k}|_V\}$ $\mathbb{C}$-basis $\forall q \in V$.

Complex Vector Bundles

Definition 60: A **complex vector bundle** is a (real) vector bundle $E \xrightarrow{\pi} M$ together with a smooth family $\{J_p\}_{p \in M}$ of complex structures $J_p \in End_{\mathbb{R}}(E_p)$, $J_p^2 = -\mathrm{id}_{E_p}$, i.e. each fibre E_p is a complex vector space.

In particular, the real rank is even and the complex rank is defined to be

$$k = \mathrm{rk}_{\mathbb{C}} E = \frac{\mathrm{rk}_{\mathbb{R}} E}{2}.$$

There exist local trivializations $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ such that

$$\Phi|_{E_p} : (E_p, J_p) \to \mathbb{C}^k$$

is complex linear for all $p \in U$.

The transition functions are smooth maps

$$g : U \cap V \to Gl(k; \mathbb{C}). \subset Gl(2k, \mathbb{R})$$

Complex Vector Bundles

Definition 60: A **complex vector bundle** is a (real) vector bundle $E \xrightarrow{\pi} M$ together with a smooth family $\{J_p\}_{p \in M}$ of complex structures $J_p \in End_{\mathbb{R}}(E_p)$, $J_p^2 = -\mathrm{id}_{E_p}$, i.e. each fibre E_p is a complex vector space.

In particular, the real rank is even and the complex rank is defined to be

$$k = \mathrm{rk}_{\mathbb{C}} E = \frac{\mathrm{rk}_{\mathbb{R}} E}{2}.$$

There exist local trivializations $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ such that

$$\Phi|_{E_p} : (E_p, J_p) \to \mathbb{C}^k$$

is complex linear for all $p \in U$.

The transition functions are smooth maps

$$g : U \cap V \to Gl(k; \mathbb{C}).$$

Vice versa: A family of such transition functions satisfying the cocycle condition defines a complex vector bundle up to isomorphism.

Complex Vector Bundles

Lemma 61: Let ∇ be a connection on the complex vector bundle $E \stackrel{\pi}{\to} M$. Then the following conditions are equivalent:

Complex Vector Bundles

Lemma 61: Let ∇ be a connection on the complex vector bundle $E \xrightarrow{\pi} M$. Then the following conditions are equivalent:

(i) $\nabla J \equiv 0$, i.e. J is parallel w.r.t. ∇ $\quad J \in \Gamma(M, End(E))$

Complex Vector Bundles

Lemma 61: Let ∇ be a connection on the complex vector bundle $E \xrightarrow{\pi} M$. Then the following conditions are equivalent:

(i) $\nabla J \equiv 0$,

(ii) For any smooth $f : M \to \mathbb{C}$ and section $\sigma : M \to E$ we have

$$\nabla(f\sigma) = df\sigma + f\nabla\sigma$$ $\Rightarrow$ usual Leibniz rule

Complex Vector Bundles

Lemma 61: Let ∇ be a connection on the complex vector bundle $E \xrightarrow{\pi} M$. Then the following conditions are equivalent:

(i) $\nabla J \equiv 0$,

(ii) For any smooth $f : M \to \mathbb{C}$ and section $\sigma : M \to E$ we have

$$\nabla(f\sigma) = df\sigma + f\nabla\sigma$$

(iii) The connection 1-form w.r.t. any complex trivialization has

the form $A \in \Omega^1(U; M(k,\mathbb{C}))$ with $M(k,\mathbb{C}) \subset M(2k;\mathbb{R})$ understood as (real) subalgebra.

Complex Vector Bundles

Lemma 61: Let ∇ be a connection on the complex vector bundle $E \xrightarrow{\pi} M$. Then the following conditions are equivalent:

(i) $\nabla J \equiv 0$,

(ii) For any smooth $f : M \to \mathbb{C}$ and section $\sigma : M \to E$ we have

$$\nabla(f\sigma) = df\sigma + f\nabla\sigma$$

(iii) The connection 1-form w.r.t. any complex trivialization has

the form $A \in \Omega^1(U; M(k, \mathbb{C}))$ with $M(k, \mathbb{C}) \subset M(2k; \mathbb{R})$ understood as (real) subalgebra.

Such a connection is called a **complex connection**.

Complex Vector Bundles

Proof of Lemma 61: (i) ⇒ (iii) ⇒ (ii) ⇒ (i)

(iii) ⇒ (ii): $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ trivialization of $\mathbb{C}$-VB

$$\nabla = \Phi^{-1} \circ (d + A) \circ \Phi$$

$$\nabla(f\sigma) = \Phi^{-1}((d+A)(f \cdot \Phi(\sigma))) \qquad \Phi(f\sigma) = f \cdot \Phi(\sigma)$$

$$= \Phi^{-1}((df)\Phi(\sigma) + f\, d(\Phi(\sigma)) + f\, A(\Phi(\sigma)))$$

$A \in \Omega^1(U, M(k,\mathbb{C}))$

$$= (df)\sigma + f(\Phi^{-1}(d+A)(\Phi(\sigma)))$$

$$= (df)\sigma + f\, \nabla\sigma$$

(ii) ⇒ (i): choose $f \equiv i$, $df = 0$

$$\nabla(J(\sigma)) = \nabla(i\sigma) = i\nabla\sigma = J(\nabla\sigma) \quad \forall \sigma \in \Gamma(U, E|_U)$$

$$\Rightarrow \nabla J \equiv 0 \quad (\text{definition of } \nabla J)$$

Left with (i) ⇒ (iii) ...

Complex Vector Bundles

Hermitian Vector Bundles

Definition 62: (i) Let $E \xrightarrow{\pi} M$ be a complex vector bundle. A Hermitian structure on E is a smooth family $\{h_p\}_{p \in M}$ of Hermitian products on E_p, i.e. $\mathbb{R}$-bilinear forms which are $\mathbb{C}$-linear in the first and $\mathbb{C}$-antilinear in the second component, satisfy $h_p(w, v) = \overline{h_p(v, w)}$ for $v, w \in E_p$ and $h_p(v, v) > 0$ if $v \neq 0$. In particular, the real part $g = \mathrm{Re}(h)$ is a euclidean structure.

Hermitian Vector Bundles

Definition 62: (i) Let $E \xrightarrow{\pi} M$ be a complex vector bundle. A Hermitian structure on E is a smooth family $\{h_p\}_{p \in M}$ of Hermitian products on E_p, i.e. $\mathbb{R}$-bilinear forms which are $\mathbb{C}$-linear in the first and $\mathbb{C}$-antilinear in the second component, satisfy $h_p(w, v) = \overline{h_p(v, w)}$ for $v, w \in E_p$ and $h_p(v, v) > 0$ if $v \neq 0$. In particular, the real part $g = \mathrm{Re}(h)$ is a euclidean structure.

(ii) A complex connection ∇ on an Hermitian vector bundle (E, h) is called **Hermitian** if it is metric w.r.t. g, provided J is orthogonal w.r.t. g.

Hermitian Vector Bundles

Definition 62: (i) Let $E \stackrel{\pi}{\to} M$ be a complex vector bundle. A Hermitian structure on E is a smooth family $\{h_p\}_{p\in M}$ of Hermitian products on E_p, i.e. $\mathbb{R}$-bilinear forms which are $\mathbb{C}$-linear in the first and $\mathbb{C}$-antilinear in the second component, satisfy $h_p(w,v) = \overline{h_p(v,w)}$ for $v, w \in E_p$ and $h_p(v,v) > 0$ if $v \neq 0$. In particular, the real part $g = \mathrm{Re}(h)$ is a euclidean structure.

(ii) A complex connection ∇ on an Hermitian vector bundle (E,h) is called **Hermitian** if it is metric w.r.t. g, provided J is orthogonal w.r.t. g.

Remark: h is determined by g and, obviously, vice versa. We have

$$h(.,) := g(.,.) + ig(.,J.)$$

(Exercise)

Hermitian Vector Bundles

Lemma 63: (i) Let (E, h) be a Hermitian vector bundle over a manifold M. Then the local trivializations $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ can be chosen to be Hermitian isomorphisms.

Hermitian Vector Bundles

Lemma 63: (i) Let (E, h) be a Hermitian vector bundle over a manifold M. Then the local trivializations $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ can be chosen to be Hermitian isomorphisms.

(ii) The curvature F of a Hermitian connection is skew symmetric w.r.t. h:

$$h(F\sigma, \tau) = -h(\sigma, F\tau).$$

Hermitian Vector Bundles

Lemma 63: (i) Let (E, h) be a Hermitian vector bundle over a manifold M. Then the local trivializations $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ can be chosen to be Hermitian isomorphisms.

(ii) The curvature F of a Hermitian connection is skew symmetric w.r.t. h:

$$h(F\sigma, \tau) = -h(\sigma, F\tau).$$

(iii) W.r.t. a trivialization described in (i) the connection 1–form and the curvature satisfy

$$A_k^\ell = -\overline{A_\ell^k} \quad \text{and } F_k^\ell = -\overline{F_\ell^k}.$$

Proof: Exercise

Almost Complex Structures

Definition 64: M smooth manifold of dimension $2n$.
(1) A complex structure J on TM is called **almost complex structure** on M.

Almost Complex Structures

Definition 64: M smooth manifold of dimension $2n$.
(1) A complex structure J on TM is called **almost complex structure** on M.
(2) A Hermitian metric on (M, J) is a Riemannian metric g such that J_p is a isometry at each $p \in M$.

Almost Complex Structures

Definition 64: M smooth manifold of dimension $2n$.
(1) A complex structure J on TM is called **almost complex structure** on M.
(2) A Hermitian metric on (M, J) is a Riemannian metric g such that J_p is a isometry at each $p \in M$.
(3) $\omega(.,.) := g(., J.) \in \Omega^2(M)$ is called **Kähler form** of (M, J, g).

Almost Complex Structures

Definition 64: M smooth manifold of dimension $2n$.
(1) A complex structure J on TM is called **almost complex structure** on M.
(2) A Hermitian metric on (M, J) is a Riemannian metric g such that J_p is a isometry at each $p \in M$.
(3) $\omega(.,.) := g(., J.) \in \Omega^2(M)$ is called **Kähler form** of (M, J, g).

Remark: (a) As seen above, (g, J) determine via $h := g + i\omega$ a Hermitian structure on TM.
(b) ω is non-degenerate: at any $p \in M$: the linear map $X \in T_pM \mapsto \omega(X, .) \in T^*M$ is an isomorphism.
Exxamples: (1) $M = \mathbb{C}^n$. Then $T_p\mathbb{C}^n \cong \mathbb{C}^n$ and for $X \in T_p\mathbb{C}^n$

$$J_p(X) := \mathrm{i}X.$$

Almost Complex Structures

Examples: (2) Let (Σ, g) be an oriented surface with a Riemannian metric g. For $X \in T_p\Sigma$ we define $J_p(X)$ by requiring, that $\{X; J_p(X)\}$ is an *oriented orthonormal basis* of $(T_p\Sigma, g_p)$. $J_p(X)$ is the counterclockwise rotated X! g defines a hermitian structure on (Σ, J).

Almost Complex Structures

Examples: (2) Let (Σ, g) be an oriented surface with a Riemannian metric g. For $X \in T_p\Sigma$ we define $J_p(X)$ by requiring, that $\{X; J_p(X)\}$ is an *oriented orthonormal basis* of $(T_p\Sigma, g_p)$. $J_p(X)$ is the counterclockwise rotated X! g defines a hermitian structure on (Σ, J).

J is unchanged if we replace g by $\lambda^2 g$ for $\lambda : \Sigma \to \mathbb{R}_+$ smooth.

Almost Complex Structures

Examples: (2) Let (Σ, g) be an oriented surface with a Riemannian metric g. For $X \in T_p\Sigma$ we define $J_p(X)$ by requiring, that $\{X; J_p(X)\}$ is an *oriented orthonormal basis* of $(T_p\Sigma, g_p)$. $J_p(X)$ is the counterclockwise rotated X! g defines a hermitian structure on (Σ, J).

J is unchanged if we replace g by $\lambda^2 g$ for $\lambda : \Sigma \to \mathbb{R}_+$ smooth.

There exists an atlas of Σ such that for each chart (U, φ, V)

$$d\varphi \circ \mathrm{i} = J \circ d\varphi.$$

(non-trivial!). The transition maps in such atlas are holomorphic functions (exercise).

Almost Complex Structures

Examples: (2) Let (Σ, g) be an oriented surface with a Riemannian metric g. For $X \in T_p\Sigma$ we define $J_p(X)$ by requiring, that $\{X; J_p(X)\}$ is an *oriented orthonormal basis* of $(T_p\Sigma, g_p)$. $J_p(X)$ is the counterclockwise rotated X! g defines a hermitian structure on (Σ, J).

J is unchanged if we replace g by $\lambda^2 g$ for $\lambda : \Sigma \to \mathbb{R}_+$ smooth.

There exists an atlas of Σ such that for each chart (U, φ, V)

$$d\varphi \circ \mathrm{i} = J \circ d\varphi.$$

(non-trivial!). The transition maps in such atlas are holomorphic functions (exercise).

(Σ, J) is called **Riemann surface**, J its **conformal structure**.

Almost Complex Structures

Examples: (2) Let (Σ, g) be an oriented surface with a Riemannian metric g. For $X \in T_p\Sigma$ we define $J_p(X)$ by requiring, that $\{X; J_p(X)\}$ is an *oriented orthonormal basis* of $(T_p\Sigma, g_p)$. $J_p(X)$ is the counterclockwise rotated X! g defines a hermitian structure on (Σ, J).

J is unchanged if we replace g by $\lambda^2 g$ for $\lambda : \Sigma \to \mathbb{R}_+$ smooth.

There exists an atlas of Σ such that for each chart (U, φ, V)

$$d\varphi \circ \mathrm{i} = J \circ d\varphi.$$

(non-trivial!). The transition maps in such atlas are holomorphic functions (exercise).

(Σ, J) is called **Riemann surface**, J its **conformal structure**. In Algebraic Geometry, (Σ, J) is called **complex curve** if $\partial\Sigma = \emptyset$.

Almost Complex Structures