Differential Geometry II

Principal Fibre Bundles

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Frame Bundles

Let $E \xrightarrow{\pi} M$ be a vector bundle over a manifold M of rank k. A (local) **frame** is a k-tupel of sections $\{\sigma_1, ..., \sigma_k\}$ on an open subset $U \subset M$, such that $\{\sigma_1(x), ..., \sigma_k(x)\}$ form a basis of E_x for any $x \in U$.

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Notice: A local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ defines a frame ob U via $\sigma_i(x) := \Phi^{-1}(x, e_i)$ for $x \in U$ and the standard basis $\{e_i\}_{i=1}^k$ of $\mathbb{R}^k$.

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The **frame bundle** $\mathcal{F}(E) \xrightarrow{\pi} M$ of E is given by

$$\mathcal{F}(E) := \coprod_{x \in M} (\{x\} \times \{(v_1, \ldots, v_k) \mid \text{ basis of } E_x\}).$$

and $\pi(x, e) = x$.

Frame Bundles

$\mathcal{F}(E)$ is a fibre bundle with fibre $Gl(k;\mathbb{R})$, the trivializations $\Psi : \pi^{-1}(U) \to U \times Gl(k;\mathbb{R})$ given by

$$\Psi((x,(v_1,...,v_k))) = (x, g(x,v))$$

where $g = (g_{ij})$ is determined by

$$v_j = \sum_{i=1}^{k} g_{ij}\sigma_i(x)$$

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If E is euclidean, complex or Hermitian one can choose orthonormal, complex or unitary frames, respectivly, and can thus define a corresponding frame bundle whose fibre is diffeomorphic to a matrix subgroup G which is $O(k)$ or $SO(k)$, $Gl(k;\mathbb{C})$ and $U(k)$, respectively.

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Remark: $O(k) \subset SO(k) \subset Gl(k;\mathbb{R})$ are subgroups and submanifolds of the open subset $Gl(k;\mathbb{R}) \subset M(k;\mathbb{R})$, $U(n) \subset Gl(k;\mathbb{C}) \subset M(k;\mathbb{C})$ is a submanifold of the open subset $Gl(k;\mathbb{C}) \subset M(k;\mathbb{C})$, the latter a linear subspace of $M(2k;\mathbb{R})$. The group operation and the inverse are differentiable maps.

Frame Bundle

Call, the corresponding group the **structure group**, G, it acts on each fibre of corresponding frame bundle $\mathcal{F}_G(E)$ from the **right**

$$R_g : (x, v) \in \mathcal{F}_G(E) \mapsto (x, vg) \in \mathcal{F}_G(E)$$

where

$$(vg)_i = \sum_{j=1}^{k} g_{ij} v_i, \qquad g_{ji} v_j$$

which satisfies $\pi((x, v)g) = x = \pi((x, v))$.

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which satisfies $\pi((x, v)g) = x = \pi((x, v))$. An affine, metric, complex, unitary connection ∇ on E gives rise to a parallel transport along any curve γ in M which is a real, orthogonal, complex or unitary iromorphism between the fibres over it.

Frame Bundles

Hence, we obtain a lift $\tilde{\gamma}$ in $\mathcal{F} := \mathcal{F}_G(E)$ and a a smooth splitting

$$\longrightarrow \quad T_{(x,v)}\mathcal{F}_G(E) = T_{(x,v)}\mathcal{F}_x \oplus T^h_{(x,v)}\mathcal{F}, \qquad \mathcal{F}_x = \pi^{-1}(x)$$

fibre

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which satisfies

$$dR_g(T^h_{(x,v)}\mathcal{F}) = T^h_{(x,v)g}\mathcal{F}$$
$$d\pi(T^h_{(x,v)}\mathcal{F}) = T_x M$$
$$dR_g(T^v_{(x,v)}\mathcal{F}_x) = T^v_{(x,v)}\mathcal{F}_x.$$

← special

$d\pi|_{T^h_{(x,v)}\mathcal{F}}$ is isomorphism ⇐ ← general for horizontal tangent spaces of fibre bundles

$R_g : \mathcal{F} \to \mathcal{F}$ $R_g(x,v) = (x, vg)$ smooth. ⇝ dR_g

$\pi : \mathcal{F} \to M$ smooth, $d\pi_{(x,v)} : T_{(x,v)}\mathcal{F} \to T_x M$

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The last condition follows from definition of $\mathcal{F}_G(E)$.

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The last condition follows from definition of $\mathcal{F}_G(E)$.

Such a splitting on $\mathcal{F}_G(E)$ also determines a corresponding affine, metric, complex or unitary connection on E.

Lie Groups

Definition 67: (i) A Lie group G is a smooth manifold (without boundary) with a group structure such that

$$(g, h) \in G \times G \mapsto gh^{-1} \in G$$

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Examples: The matrix subgroups $O(n), SO(n), U(n)$ are Lie groups.

Lie Groups

Proposition 68: Let G be a Lie group. Then its tangent bundle is trivialized via

$$g \in G, X \in T_e G \mapsto d_g L_g(X)$$

where $L_g : G \to G$, $L_g(H) = gh$, the left action of G on itself, is smooth by definition. Denote the corresponding vector field by $\tilde{X}$.

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$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

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$T_e G$ is called the **Lie algebra** of G and denoted by $\underline{g}$. (= $\mathfrak{g}$ in literature)

Lie Groups

Proof: (i) We have $X \in \mathfrak{g}$

$$
\begin{aligned}
(dL_g(\tilde{X}))_h &= d_{g^{-1}h}L_g(\tilde{X}_{g^{-1}h}) = d_{g^{-1}h}L_g(d_e L_{g^{-1}h}(X)) \\
&= d_e(L_g \circ L_{g^{-1}h})(X) = d_e L_h(X) \\
&= \tilde{X}_h.
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Since L_g is a diffeomorphism theis is equal to

$$= [dL_g(\tilde{X}), dL_g(\tilde{Y})]_e = [\tilde{X}, \tilde{Y}]_e$$

since $\tilde{X}, \tilde{Y}$ are left-invariant. □

Principal Fibre Bundles

Definition 69: Let G be a Lie group. A **principal G–bundle** over a manifold M is fibre bundle, $P \xrightarrow{\pi} M$, together with a smooth right G-action which preserves the fibres and the trivializations $\Phi : \pi^{-1}(U) \to U \times G$ can be chosen so that for all $x \in U, h \in G$, $\Phi^{-1}(x, hg) = \Phi^{-1}(x, h)g$

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Remark: A smooth right G-action on a manifold P is a smooth map $\mu : P \times G \to P$ such that with $\mu_g : P \to P$, $\mu_g(p) = \mu(p, g) =: pg$ for all $g, h \in G$

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Examples: (i) The frame bundles of vector bundles (real, complex, euclidean, oriented or unitary) are principal fibre bundles with the group G being provided by the structure group.

(ii) The Hopf fibration is a principal fibre bundle with $G = S^1 = U(1) = SO(2)$.

Connections

Definition 70: Let G be a Lie group. A **connection** on a principal G-bundle $P \xrightarrow{\pi} M$ is a smooth family $\{T_p^h P\}_{p \in P}$ of subspaces of $T_p P$ such that:

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$\mu : P \times G \to P$, $\mu(p, .) : G \to P$ smooth

$d_e \mu(p, .) : \underline{g} \to T_p \pi^{-1}(\pi(p))$ is an isomorphism and we define

$$A_p := d_e \mu(p, .)^{-1} \circ \tilde{A}_p : T_p P \to \underline{g}$$

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$\Rightarrow T_p^h P = \ker A_p$

Condition (ii) translates to

$$\mathrm{Ad}_g \circ A_{pg} = A_p$$

stille

$\tilde{A}_{pg} = F_{p,g} \circ \tilde{A}_p \circ (d\mu_g)^{-1}$

$F_{p,g} : T_pP \to T_{pg}P$ isomorphism

claim: $F_{p,g} = d\mu_g$

where $\alpha_g : G \to G$, $\alpha_g(h) = ghg^{-1}$ is the **conjugation**, $\mathrm{Ad}_g := d_e\alpha_g : \underline{g} \to \underline{g}$ the **adjoint representation** of G.

Proof: G-invariance of $T_p^h P$. $d_p r_g (T_p^h P) = T_{pg}^h P$

$$\Leftrightarrow \quad \tilde{A}_{pg} \circ d_p r_g = \tilde{A}_p \qquad (*)$$

since $T_p^h P = \ker \tilde{A}_p$ $\quad \tilde{A}_{pg}\left(d_p r_g (T_p^h P)\right) = \tilde{A}_p (T_p^h P) = 0$

showing $(\Leftarrow)$.

$(\Rightarrow)$ check that using Lie Alg.

wand to show $\quad (*) \Leftrightarrow \mathrm{Ad}_g \circ A_{pg} = A_p$

will be corrected on June 9

$$\Leftrightarrow \quad d_e \alpha_g \circ d_e \mu(pg,\cdot)^{-1} \circ \tilde{A}_{pg} = d_e \mu(p,\cdot)^{-1} \tilde{A}_p$$

$$= d_e \mu(p,\cdot)^{-1} \tilde{A}_{pg} \circ d_p r_g$$

$$\mu(pg,h) = \mu(p,gh) = (\mu(p,\cdot) \circ L_g)(h)$$

$$d\mu(pg,\cdot)^{-1} = dL_g^{-1} \circ d\mu(p,\cdot)^{-1} \quad \to \underline{\text{finish}}.$$

Associated Bundles

A group homomorphism $\rho : G \to \mathrm{Aut}(V)$ for a $\mathbb{K}$–vector space V is called a $\mathbb{K}$**–representation of** G. Let $P \stackrel{\pi_P}{\to} M$ be a principal G–bundle. The **associated vector bundle**

$$P \times_\rho V := (P \times V / \sim) \stackrel{\pi}{\to} M$$

$= P \times V / G$ is a quotient by G-action determined by ρ

where $(p, v) \sim (pg, \rho(g^{-1})v)$ for all $p \in P, v \in V, g \in G$ and $\pi([p, v]) := \pi_P(p)$.

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If V is euclidean, Hermitian or carries a (Lie) algebra structure and $\rho(g)$ preserves it, then so does $P \times_\rho V$. (Exercise)

Covariant Exterior Derivative

We denote by $\underline{\mathbf{g}} := P \times_{\mathrm{Ad}} \underline{g}$ the associated Lie algebra bundle.

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We define $\Omega^k(M; \underline{\mathbf{g}})$ to consist of smooth sections of $\Lambda^k(M) \otimes \underline{\mathbf{g}}$.
We have

$\tilde{X}_p \lrcorner \alpha_p = 0 \quad \forall X \in \underline{g}$

$$\Omega^k(M; \underline{\mathbf{g}}) = \{\alpha \in \Omega^k(P, \underline{g}) \mid \alpha_p|_{T_p \pi^{-1}(\pi(p))} = 0, \mathrm{Ad}_g \circ \alpha_{pg} = \alpha_p\}$$

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We denote by $\underline{\mathbf{g}} := P \times_{\mathrm{Ad}} \underline{g}$ the associated Lie algebra bundle.

We define $\Omega^k(M; \underline{\mathbf{g}})$ to consist of smooth sections of $\Lambda^k(M) \otimes \underline{\mathbf{g}}$. We have

$$\Omega^k(M; \underline{\mathbf{g}}) = \{\alpha \in \Omega^k(P, \underline{g}) \mid \alpha_p|_{T_p\pi^{-1}(\pi(p))} = 0, \mathrm{Ad}_g \circ \alpha_{pg} = \alpha_p\} \leftarrow$$

$\Rightarrow \tilde{X} \lrcorner \alpha_\alpha = 0 \quad \forall X \in \underline{g}$

For a connection A we define the associated **covariant exterior derivative** $D_A : \Omega^k(M; \underline{\mathbf{g}}) \to \Omega^{k+1}(M; \underline{\mathbf{g}})$ by

$$D_A\omega := \underline{d\omega} + [A, \omega]$$

Exercise: Show that $D_A\omega \in \Omega^{k+1}(M; \underline{\mathbf{g}})$.

Curvature

The space of connections $\mathcal{C}(P)$ is an affine space over $\Omega^1(M, \underline{\mathbf{g}})$.

$\forall X \in \underline{g} \quad A_p(\tilde{X}_p) = X, \quad A'_p(\tilde{X}_p) = X \Rightarrow (A - A')_p(\tilde{X}_p) = 0$

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Definition 71: Let $A \in \mathcal{C}(P)$ be a connection. The **curvature of** A is the 2-form $F \in \Omega^2(P, \underline{g})$ given by

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(ii) 2nd Bianchi identity: $D_A F_A = 0$.

(iii) Let $X, Y \in T_x M$ and X^h, Y^h two horizontal vector fields on P in a neighbourhood of $p \in \pi^{-1}(x)$ with $d_p\pi(X^h) = X$ and $d_p\pi(Y^h) = Y$. Then

$$\widetilde{F_A(X, Y)}_p = [X^h, Y^h]_p. - [X,Y]^h_p$$

← Same mistake as previously!