Differential Geometry II

Principal Fibre Bundles II

Connections and Curvature

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Connections on Principal Fibre Bundles

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$$\Rightarrow \quad \tilde{A}_{pg} \circ d\mu_g = d\mu_g \circ \tilde{A}_p$$

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$d_e\mu(p, .) : \underline{g} \to T_p \pi^{-1}(\pi(p))$ is an isomorphism and we define

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Condition (ii) translates to

$$\mathrm{Ad}_{g^{-1}} \circ A = \underline{\mu_g^* A}$$

where $\alpha_g : G \to G$, $\alpha_g(h) = ghg^{-1}$ is the **conjugation**, $\mathrm{Ad}_g := d_e\alpha_g : \underline{g} \to \underline{g}$ the **adjoint representation** of G.

$\tilde{A}_p \quad T_p P \longrightarrow T_p P_x \qquad x = \pi(p),\ P_x = \pi^{-1}(x)$

- linear, $\tilde{A}_p^2 = 0$ projection
- $d_p\mu_g(\operatorname{Ker}\tilde{A}_p) = \operatorname{Ker}\tilde{A}_{pg}$ (by (ii))

$\operatorname{Ker}\tilde{A}_p = \{v_1, \ldots, v_m\}$, $\operatorname{Im}\tilde{A}_p = \{w_1, \ldots, w_n\}$

basis $\tilde{v}_j := d\mu_g(v_j)$, $\tilde{w}_j := d\mu_g(w_j)$

$A_{pg}\tilde{v}_j = 0$ by (ii) $\qquad \tilde{A}_p(w_j) = w_j \quad \underline{\tilde{A}_p \text{ proj}}$

- $d\mu_g(\underbrace{\operatorname{Im}\tilde{A}_p}_{T_p P_x}) = \operatorname{Im}\tilde{A}_{pg} = T_{pg}P_x$

$\Rightarrow \tilde{A}_{pg}(\tilde{w}_j) = \tilde{w}_j \quad \tilde{A}_{pg}$ proj

$\Rightarrow A_{pg} \cdot d\mu_g(v_i) = 0 = d\mu_g \cdot A_p(v_i)$

$$\& \quad A_{pg} \cdot d\mu_g(w_1) = d\mu_g(w_\cdot) = d\mu_g \cdot A_p(w_1)$$

$$A_{pg} = d_e\mu(pg,\cdot)^{-1} \cdot \tilde{A}_{pg} = d_e\mu(pg,\cdot)^{-1} \cdot d\mu_g \cdot \underline{\tilde{A}_p} \cdot d\mu_g^{-1}$$

$$\Rightarrow A_{pg} \cdot d\mu_g = d_e\mu(pg,\cdot)^{-1} \cdot d\mu_g \cdot \underbrace{d_e\mu(p,\cdot) \cdot \underbrace{A_p}_{= X \in \mathfrak{g}}}_{= \tilde{X}_p}$$

$$\| $$

$$(\mu_g^* A)_p =$$

pick $\gamma : (-\varepsilon, \varepsilon) \to G \quad \gamma(0) = e \quad \dot\gamma(0) = X \in T_e G$

$$\tilde{X}_p = [p\gamma], \quad d\mu_g(\tilde{X}_p) \overset{!}{=} [p\gamma g] = [pg(g^{-1}\gamma g)]$$

$$d_e\mu(pg,\cdot)^{-1}(d\mu_g(\tilde{X}_p)) = [g^{-1}\gamma g] = Ad_{g^{-1}}(X$$

Covariant Exterior Derivative

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We have $\forall X \in \mathfrak{g}$ $\quad \alpha_p(\tilde{X}, \ ) = i_{\tilde{X}} \alpha_p$

$$\Omega^k(M;\underline{\mathbf{g}}) = \{\alpha \in \Omega^k(P,\underline{g}) \mid \alpha_p|_{\cancel{T_p\pi^{-1}(\pi(p))}} = 0, \mathrm{Ad}_g \circ \alpha_{pg} = \alpha_p\}$$

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$$\forall X \in \underline{g} \quad \alpha_p(\tilde{X}, \cdot) = 0$$

$$\Omega^k(M; \underline{\mathbf{g}}) = \{\alpha \in \Omega^k(P, \underline{g}) \mid \alpha_p\cancel{|_{T_p \pi^{-1}(\pi(p))}} = 0, \mathrm{Ad}_g \circ \alpha_{pg} = \alpha_p\}$$

For a connection A we define the associated **covariant exterior derivative** $D_A : \Omega^k(M; \underline{\mathbf{g}}) \to \Omega^{k+1}(M; \underline{\mathbf{g}})$ by

$$D_A \omega := d\omega + [A, \omega]$$

Exercise: Show that $D_A \omega \in \Omega^{k+1}(M; \underline{\mathbf{g}})$.

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Definition 71: Let $A \in \mathcal{C}(P)$ be a connection. The **curvature of** A is the 2-form $F_A \in \Omega^2(P, \underline{g})$ given by

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Lemma 72: (i) F_A vanishes on tangent vectors tangent to the fibre, i.e.

$$F(\tilde{X}, .) = 0$$

for all $X \in \underline{g}$. In particular, $F \in \Omega^2(M; \underline{\mathbf{g}})$.

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for all $X \in \underline{g}$. In particular, $F_A \in \Omega^2(M; \underline{\mathbf{g}})$: $\mu_g^* F_A = Ad_g \cdot F_A$!

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Curvature

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(ii) 2nd Bianchi identity: $D_A F_A = 0$.

(iii) Let $X, Y \in T_x M$ and X^h, Y^h two horizontal vector fields on P in a neighbourhood of $p \in \pi^{-1}(x)$ with $d_p\pi(X^h) = X$ and $d_p\pi(Y^h) = Y$. Then

$$F_A(X, Y)_p = [X^h, Y^h]_p - [X, Y]^h_p \quad !$$

Proof (i) $F_p(\tilde{X}, Y) = 0 \quad \forall X \in \mathfrak{g} \; \forall Y \in T_p P$

(extended to vector fields)

$$F(\tilde{X}, Y) = \tilde{X}(A(Y)) - \underbrace{Y(A(\tilde{X}))}_{=0} - A([\tilde{X}, Y]) + \frac{1}{2}\left([A(\tilde{X}), A(Y)] - [A(Y), A(\tilde{X})]\right) = \clubsuit$$

$A(\tilde{X}) = X$ (def of A)

- $Y = \tilde{z}, \; z \in \mathfrak{g}$ $\qquad \tilde{X}(A(\tilde{z})) = 0 \quad A(\tilde{z}) \equiv z$

$\qquad [\tilde{X}, \tilde{z}] = \widetilde{[X, z]}$

$$\clubsuit = 0 - 0 - A(\widetilde{[X, z]}) + [A(\tilde{X}), A(\tilde{z})]$$
$$= -[X, z] + [X, z] = 0$$

$Y = W^h$, W vf on M, $Y_{r_g} = d\mu_g(Y_p)$ $\quad Y_{\phi_t}$ ← flow of $\tilde{X}$

$$\Rightarrow [\tilde{X}, Y]_p = \pm(\mathcal{L}_{\tilde{X}} Y)_p = \pm \frac{d}{dt}\Big|_{t=0} \left(d\phi_{\tilde{X}} \; Y_{\phi_{\tilde{X}}(p)}\right)$$

Let φ_X^t be flow of $\tilde{X}$ on G $(= \exp(tX))$

$\Rightarrow \phi_{\tilde{X}}^t(p) = p\,\varphi_X^t(e)$, $d\phi_{\tilde{X}}^{-t} = d\mu_{\varphi_X^t(e)}$ $\leftarrow \varphi_X^{-t}(e)$

$$\Delta = \frac{d}{dt}\Big(\underbrace{d_{p\,\varphi_X^t(e)}\mu_{\varphi_X^{-t}(e)}\big(\Upsilon_{p\,\varphi_X^t(e)}\big)}_{=\Upsilon_p} \Big) = 0$$

$$\cancel{A} = \cancel{\tilde{X}(A(\Upsilon))} - \underbrace{\Upsilon(A(\tilde{X}))}_{=0} - A([\tilde{X},\Upsilon])$$

$$+ \cancel{[A(\tilde{X}), A(\Upsilon)]} \qquad A(\Upsilon) \equiv 0 \;!$$

$$= -A([\tilde{X},\Upsilon]) = 0 \quad (\mathcal{D}(i))$$

(ii) See below!

Relation to Vector Bundles

Let G be a Lie group, $P \xrightarrow{\pi} M$ be a principal G–bundle and A be a connection on P. Let $\rho : G \to Aut(V)$ be a finite-dimensional representation of G on a $\mathbb{K}$-vector space V $(\mathbb{K} = \mathbb{R}, \mathbb{C})$.

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Then A induces a connection $\nabla = \nabla^{A,\rho}$ on the associated bundle $E = P \times_\rho V$ as follows: A smooth section $\sigma : U \to E$ is uniquely determined by $\tilde{\sigma} : \pi^{-1}(U) \to V$ such that $\tilde{\sigma}(pg) = \rho(g^{-1})\tilde{\sigma}(p)$.

Then $\sigma(x) = [p, \tilde{\sigma}(p)]$ for $p \in \pi^{-1}(x)$

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$$D_A\tilde{\sigma} := d\tilde{\sigma} + \rho_*(A)(\tilde{\sigma}) \in \Omega^1(P; V)$$

vanishes on vertical tangent vectors and descends to $\nabla\sigma \in \Omega^1(M; E)$.

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$= T_{id_E} Aut(V)$

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The curvature of ∇ is given by

$$F^\nabla = \rho_*(F_A).$$

The natural projection $\pi : P \times V \to E = P \times_\rho V$ to the G–quotient is smooth.

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$$T^h_{[p,v]}E = \underline{d_{p,v}\pi}(T^h_p P)$$

for the connection on P and its induced connection on E.

Proof of Lemma 7.2 (ii)

$$D_A F_A \overset{\text{Def.}}{=} \underbrace{d(dA}_{=0} + \tfrac{1}{2}[A,A]) + [A, (dA + \tfrac{1}{2}[A,A])]$$

$$= \tfrac{1}{2}\big([dA, A] - [A, dA]\big) + [A, dA] + \tfrac{1}{2}[A,[A,A]]$$

$$[A, dA] = -[dA, A] \qquad\qquad A = \sum A_i\, dx^i$$

in coord. on M

$$= \frac{1}{2} \sum_{1 \le i < j < k \le n = \dim M} \big([A_i,[A_j,A_k]] + [A_j,[A_k,A_i]] + [A_k,[A_i,A_j]]\big)\, dx^i \wedge dx^j \wedge dx^k$$

$$= 0$$

The Quaternionic Hopf Bundle