Differential Geometry II

Characteristic Classes

Klaus Mohnke

June 11, 2020

The Quaternionic Hopf Bundle

Let

$$S^7 := \{A \in M(2;\mathbb{C}) \mid \mathrm{Trace}(\overline{A}^T A) = \cancel{2}\,1\} \subset M(2;\mathbb{C}) \cong \mathbb{C}^4.$$

$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ $\quad \mathrm{Trace}(\overline{A}^T A) = |a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2$

The Quaternionic Hopf Bundle

Let

$$S^7 := \{A \in M(2;\mathbb{C}) \mid \mathrm{Trace}(\overline{A}^T A) = 2\} \subset M(2;\mathbb{C}) \cong \mathbb{C}^4.$$

The Lie group $SU(\overset{\simeq S^3}{2}) := \{g \in M(2;\mathbb{C}) | \overline{g}^T g = \mathbf{E}_2, \det g = 1\}$ is acting on it (from the right) via $A \mapsto Ag$ and

$$S^7/SU(2) \cong S^4$$

$\mathrm{Trace}(\overline{Ag}^T Ag) = \mathrm{Trace}(\overline{g}^T \overline{A}^T A g) = \mathrm{Trace}(g^{-1}\overline{A}^T A g) = \mathrm{Trace}(\overline{A}^T A) = 1.$

are diffeomorphic.

Proof: $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in S^7$ $\quad \exists g \in SU(2): a_1 \in \mathbb{R}, a_2 = 0.$

wlog. $(a_1, a_2) \neq (0,0)$ Let $(a,b) := \frac{1}{\sqrt{|a_1|^2 + |a_2|^2}}(a_1, a_2)$ $\quad |a|^2 + |b|^2 = 1.$

$g := \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix}$ $\quad \bar{g}^T g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix} = E_2$

$Ag = \begin{pmatrix} a_1\bar{a} + a_2\bar{b} & 0 \\ * & * \end{pmatrix} = \begin{pmatrix} \tilde{a}_{11} & 0 \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix}$ $\quad \tilde{a}_{11} \in \mathbb{R}$ $\quad \tilde{a}_{11}^2 + |\tilde{a}_{21}|^2 + |\tilde{a}_{22}|^2 = 1$

$\subset \mathbb{R} \times \mathbb{C} \times \mathbb{C} \simeq \mathbb{R}^5$

The Quaternionic Hopf Bundle

Let

$$S^7 := \{A \in M(2;\mathbb{C}) \mid \mathrm{Trace}(\overline{A}^T A) = 2\} \subset M(2;\mathbb{C}) \cong \mathbb{C}^4.$$

The Lie group $SU(2) := \{g \in M(2;\mathbb{C}) | \overline{g}^T g = \mathbf{E}_2, \det g = 1\}$ is acting on it (from the right) via $A \mapsto Ag$ and

$$S^7/SU(2) \cong S^4$$

are diffeomorphic.

The quotient map $S^7 \xrightarrow{\pi} S^4$ is a principal $SU(2)$–bundle. (Exercise)

Remark: This would be true for any free action of a compact Lie group G on a manifold M: $M \xrightarrow{\pi} M/G$ is a principal G-bundle over the manifold M/G

Ehresmann's Theorem

The Quaternionic Hopf Bundle

Let

$$S^7 := \{A \in M(2;\mathbb{C}) \mid \mathrm{Trace}(\overline{A}^T A) = 2\} \subset M(2;\mathbb{C}) \cong \mathbb{C}^4.$$

The Lie group $SU(2) := \{g \in M(2;\mathbb{C}) | \overline{g}^T g = \mathbf{E}_2, \det g = 1\}$ is acting on it (from the right) via $A \mapsto Ag$ and

$$S^7/SU(2) \cong S^4$$

are diffeomorphic.

The quotient map $S^7 \xrightarrow{\pi} S^4$ is a principal $SU(2)$–bundle. (Exercise)

$T_p^h S^7 := (T_p \pi^{-1}([p]))^\perp$ defines a connection of the principal $SU(2)$–bundle. (Exercise)

The Quaternionic Hopf Bundle

Let

$$S^7 := \{A \in M(2;\mathbb{C}) \mid \mathrm{Trace}(\overline{A}^T A) = 2\} \subset M(2;\mathbb{C}) \cong \mathbb{C}^4.$$

The Lie group $SU(2) := \{g \in M(2;\mathbb{C}) | \overline{g}^T g = \mathbf{E}_2, \det g = 1\}$ is acting on it (from the right) via $A \mapsto Ag$ and

$$S^7/SU(2) \cong S^4$$

are diffeomorphic.

The quotient map $S^7 \xrightarrow{\pi} S^4$ is a principal $SU(2)$–bundle. (Exercise)

$T_p^h S^7 := (T_p \pi^{-1}([p]))^\perp$ defines a connection of the principal $SU(2)$–bundle. (Exercise)

Chern Classes: Axioms

Definition 73: The (rational) Chern classes are a sequence $(c_k)_{k=0}^{\infty}$ which assign to each complex vector bundle $E \xrightarrow{\pi} M$ over a smooth manifold an element $C_k(E) \in H^{2k}_{DR}(M)$ with $c_0(E) = 1$ for all E, which satisfies the following axioms:

Chern Classes: Axioms

Definition 73: The (rational) Chern classes are a sequence $(c_k)_{k=0}^{\infty}$ which assign to each complex vector bundle $E \xrightarrow{\pi} M$ over a smooth manifold an element $C_k(E) \in H^{2k}_{DR}(M)$ with $c_0(E) = 1$ for all E, which satisfies the following axioms:

Chern Classes: Axioms

Definition 73: The (rational) Chern classes are a sequence $(c_k)_{k=0}^{\infty}$ which assign to each complex vector bundle $E \xrightarrow{\pi} M$ over a smooth manifold an element $C_k(E) \in H^{2k}_{DR}(M)$ with $c_0(E) = 1$ for all E, which satisfies the following axioms:

$\forall\ E \xrightarrow{\pi} N$

(ii) For all~~e~~ smooth $f : M \to N$: $c_k(f^*E) = f^*c_k(E)$

Chern Classes: Axioms

Definition 73: The (rational) Chern classes are a sequence $(c_k)_{k=0}^{\infty}$ which assign to each complex vector bundle $E \xrightarrow{\pi} M$ over a smooth manifold an element $C_k(E) \in H^{2k}_{DR}(M)$ with $c_0(E) = 1$ for all E, which satisfies the following axioms:

(i) For alle smooth $f : M \to N$: $c_k(f^*E) = f^*c_k(E)$
(ii) $c_k(E \oplus F) = \sum_{j=0}^{k} c_j(E) \wedge c_{k-j}(F)$.

Chern Classes: Axioms

Definition 73: The (rational) Chern classes are a sequence $(c_k)_{k=0}^{\infty}$ which assign to each complex vector bundle $E \xrightarrow{\pi} M$ over a smooth manifold an element $C_k(E) \in H^{2k}_{DR}(M)$ with $c_0(E) = 1$ for all E, which satisfies the following axioms:

(i) For alle smooth $f : M \to N$: $c_k(f^*E) = f^*c_k(E)$

(ii) $c_k(E \oplus F) = \sum_{j=0}^{k} c_j(E) \wedge c_{k-j}(F)$.

(iii) For the tautological bundle $H \xrightarrow{\pi} \mathbb{C}P^1$ we have

$c = 1 + c_1 + c_2 + c_3 + \dots$

$$c(\overset{H}{\cancel{E}}) = 1 - [\omega]$$

means

$c_0(E) = 1$

$c_1(E) = -[\omega]$

$c_k(E) = 0 \quad \forall k > 1$

where $\omega \in \Omega^2(\mathbb{C}P^1)$ with

$$\int_{\mathbb{C}P^1} \omega = 1.$$

(iv) For $k > rk_{\mathbb{C}}(E)$ the class vanishes: $c_k(E) = 0$.

(iii) + (iv) $\Leftrightarrow$

$\forall\ H \xrightarrow{\pi} \mathbb{C}P^k$ tautological bdle

$c(H) = 1 - [\omega]$

ω = Fubini-Study form

$\int_{\mathbb{C}P^1} \omega = 1.$

Chern Classes: Axioms

Theorem 74: (1) The Chern classes exist and are uniquely defined by the axioms.

Chern Classes: Axioms

Theorem 74: (1) The Chern classes exist and are uniquely defined by the axioms.

(2) c_k are integer valued. In particular for any complex bundle $E \xrightarrow{\pi} M$ over a closed, oriented manifold M of dimension $\dim M = 2k$ we have

$$\int_M c_k(E) \in \mathbb{Z}.$$

Chern Classes: Axioms

Theorem 74: (1) The Chern classes exist and are uniquely defined by the axioms.

(2) c_k are integer valued. In particular for any complex bundle $E \xrightarrow{\pi} M$ over a closed, oriented manifold M of dimension $\dim M = 2k$ we have

$$\int_M c_k(E) \in \mathbb{Z}.$$

The Proof is omitted since it should be rather discussed in the context of a lecture on Algebraic Topology or the Atiyah-Singer index theorem.

The Chern-Weil Construction

Let $(E, \cancel{h}) \xrightarrow{\pi} M$ be a complex vector bundle over a smooth manifold, ∇ a complex connection. Then we define the **Chern-Weil forms** $c_k(E, \nabla)$ to be the coefficients

$\in \Omega^{2k}(M, \mathbb{R})$

$$\det\left(\underbrace{\frac{\mathrm{i}t}{2\pi}F^{\nabla}}_{\in\, \Omega^2(M,\, \mathrm{End}(E))} + \mathrm{id}_E\right) = 1 + c_1(E, \nabla)t^{\cancel{2}} + c_2(E, \nabla)t^{\cancel{4}\,2} + \ldots$$

Theorem 75: (1) The Chern forms are real closed forms $c_k(E, \nabla) \in \Omega^{2k}(M; \mathbb{R})$.

The Chern-Weil Construction

Let $(E, h) \xrightarrow{\pi} M$ be a complex vector bundle over a smooth manifold, ∇ a complex connection. Then we define the **Chern-Weil forms** $c_k(E, \nabla)$ to be the coefficients

$$\det\left(\frac{\mathrm{i}t}{2\pi}F^{\nabla} + \mathrm{id}_E\right) = 1 + c_1(E, \nabla)t^2 + c_2(E, \nabla)t^4 + \ldots$$

Theorem 75: (1) The Chern forms are real closed forms $c_k(E, \nabla) \in \Omega^{2k}(M; \mathbb{R})$.

(2) The class $c_k(E) := [c_k(E, \nabla)] \in H^{2k}_{DR}(M; \mathbb{R})$ does not depend on ∇.

The Chern-Weil Construction

Let $(E,h) \xrightarrow{\pi} M$ be a complex vector bundle over a smooth manifold, ∇ a complex connection. Then we define the **Chern-Weil forms** $c_k(E,\nabla)$ to be the coefficients

$$\det\left(\frac{it}{2\pi}F^{\nabla} + \mathrm{id}_E\right) = 1 + c_1(E,\nabla)t^{\overset{1}{\cancel{2}}} + c_2(E,\nabla)t^{\overset{2}{\cancel{4}}} + \ldots$$

Theorem 75: (1) The Chern forms are real closed forms $c_k(E,\nabla) \in \Omega^{2k}(M;\mathbb{R})$.

(2) The class $c_k(E) := [c_k(E,\nabla)] \in H^{2k}_{DR}(M;\mathbb{R})$ does not depend on ∇.

(3) The sequence c_k are the (rational) Chern classes, i.e. it satisfies the axioms.

The Chern-Weil Construction

Computation of the first Chern classes:

The Chern-Weil Construction

Computation of the first Chern classes:

$$c_1(E, \nabla) = \frac{\mathrm{i}}{2\pi} \mathrm{Trace} F^{\nabla}.$$

The Chern-Weil Construction

Computation of the first Chern classes:

$$c_1(E, \nabla) = \frac{\mathrm{i}}{2\pi} \mathrm{Trace} F^{\nabla}.$$

and

$$c_2(E, \nabla) = \frac{1}{8\pi^2} (\mathrm{Trace}(F^{\nabla} \wedge F^{\nabla}) - \mathrm{Trace} F^{\nabla} \wedge \mathrm{Trace} F^{\nabla})$$

The Chern-Weil Construction

Computation of the first Chern classes:

$$c_1(E,\nabla) = \frac{\mathrm{i}}{2\pi}\mathrm{Trace} F^\nabla.$$

and

$$c_2(E,\nabla) = \frac{1}{8\pi^2}\left(\mathrm{Trace}(F^\nabla \wedge F^\nabla) - \mathrm{Trace}F^\nabla \wedge \mathrm{Trace}F^\nabla\right)$$

Ex.

Lemma 76: There is a sequence of homogeneous polynomials (p_k) of degree k with rational coefficients, such that

$$c_k(E,\nabla) = p_k(s_1,\dots,s_k)$$

where

$$s_\ell = \mathrm{Trace}\Big(\big(\frac{\mathrm{i}}{2\pi}F^\nabla\big)^\ell\Big)$$

$\overset{\neq 0}{c}\, s_k + a_{(k-1)1} s_1 s_{k-1} + a_{11(k-2)} s_1^2 s_{k-2} + \dots$

where the ℓ-th variable has degree ℓ and $p_k(0,0,\dots,1) \neq 0$.

$\Rightarrow$ for (1) & (2) of Theorem 75 it is enough to show this for s_ℓ !

The Chern-Weil Construction

Lemma 77: We have for $\alpha \in \Omega^k(M, End(E)), \beta \in \Omega^\ell(M; End(E))$ that

% that k is not necess.
% the rank of E :)

$$d(\text{Trace}(\underbrace{\alpha \wedge \beta}_{\in \Omega^{k+\ell}(M, End(E))})) = \text{Trace}(D\alpha \wedge \beta + (-1)^k \alpha \wedge D\beta).$$

D exterior covariant derivative induced by connection ∇.

Proof: Let $A \in \Omega^1(U, M(k,\mathbb{C}))$ connection 1-form of ∇ w.r.t. a trivialization $\Phi: \pi^{-1}(U) \to U \times \mathbb{C}^k$ over open $U \subset M$. $\alpha, \beta \in \Omega^*(M, M(k,\mathbb{C}))$

$d(\text{Trace}(\alpha \wedge \beta)) = \text{Trace}(D(\alpha \wedge \beta)) = \text{Trace}(D\alpha \wedge \beta + (-1)^k \alpha \wedge D\beta)$

need to show $\boxed{d(\text{Trace}(\gamma)) = \text{Trace}(D(\gamma))}$ for $\gamma \in \Omega^\ell(M, End(E))$

r.h.s: $D\gamma = d\gamma + [A, \gamma] = d\gamma + \sum_{i,I} [A_i, \gamma_I] dx^i \wedge dx^I$

where $A = \sum A_i dx^i$, $\gamma = \sum \gamma_I dx^I$ $\qquad = \text{Trace}(\sum_{i,I} [A_i, \gamma_I]) dx^i dx^I$

$I = \{1 \le i_1 < \ldots < i_\ell \le n\}$

$dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_\ell}$; $A_i, \gamma_I \in M(k, \mathbb{R})$ $\qquad = 0$

The Chern-Weil Construction

Applying Lemma 77 inductively we conclude

$$d(s_\ell(E,\nabla)) = \left(\frac{i}{2\pi}\right)^\ell \mathrm{Trace}(\underbrace{DF^\nabla}_{=0} \wedge \ldots \wedge F^\nabla + \ldots + F^\nabla \wedge \ldots \wedge \underbrace{DF^\nabla}_{=0}) = 0.$$

2nd Bianchi

The Chern-Weil Construction

Applying Lemma 77 inductively we conclude

$$d(s_\ell(E,\nabla)) = \mathrm{Trace}(DF^\nabla \wedge ... \wedge F^\nabla + ...F^\nabla \wedge ... \wedge DF^\nabla) = 0. \Rightarrow (1)$$

Let ∇^0, ∇^1 be two complex connections on E. Then

$$\nabla^1 = \nabla^0 + \alpha$$

for $\alpha \in \Omega^1(M; End_{\mathbb{C}}(E))$. Denote by $\nabla^t = \nabla^0 + t\alpha$ family of connections $t \in [0,1]$. Then $F^t = F^0 + tD^0\alpha + t^2/2[\alpha,\alpha]$.

The Chern-Weil Construction

Applying Lemma 77 inductively we conclude

$$d(s_\ell(E,\nabla)) = \mathrm{Trace}(DF^\nabla \wedge ... \wedge F^\nabla + ...F^\nabla \wedge ... \wedge DF^\nabla) = 0.$$

Let ∇^0, ∇^1 be two complex connections on E. Then

$$\nabla^1 = \nabla^0 + \alpha$$

for $\alpha \in \Omega^1(M; End(E))$. Denote by $\nabla^t = \nabla^0 + t\alpha$ family of connections $t \in [0,1]$. Then $F^t = F^0 + tD^0\alpha + t^2/2[\alpha,\alpha]$.

$$\frac{dF^t}{dt} = D^0\alpha + [t\alpha,\alpha] = D^t\alpha.$$

The Chern-Weil Construction

Applying Lemma 77 inductively we conclude

$$d(s_\ell(E,\nabla)) = \mathrm{Trace}(DF^\nabla \wedge \ldots \wedge F^\nabla + \ldots F^\nabla \wedge \ldots \wedge DF^\nabla) = 0.$$

Let ∇^0, ∇^1 be two complex connections on E. Then

$$\nabla^1 = \nabla^0 + \alpha$$

for $\alpha \in \Omega^1(M; End(E))$. Denote by $\nabla^t = \nabla^0 + t\alpha$ family of connections $t \in [0,1]$. Then $F^t = F^0 + tD^0\alpha + t^2/2[\alpha,\alpha]$.

$$\frac{dF^t}{dt} = D^0\alpha + [t\alpha,\alpha] = D^t\alpha.$$

Hence

$$\Omega^{2\ell}(M; End(E)) \ni \frac{d}{dt}s_\ell(E,\nabla^t) = \mathrm{Trace}(D^t\alpha \wedge F^t \ldots \wedge F^t + \ldots)$$

$$= d(\underbrace{\mathrm{Trace}(\alpha \wedge F^t \wedge \ldots \wedge F^t + \ldots)}_{\in \Omega^{2\ell-1}(M,\mathbb{R}) \text{ smoothly depending on } t}).$$

The Chern-Weil Construction

We obtain

$$s_\ell(E,\nabla^1) - s_\ell(E,\nabla^0) = d\Big(\int_0^1 \mathrm{Trace}(\alpha \wedge F^t \wedge \ldots \wedge F^t + \ldots)\,dt\Big)$$

Still need to show, that $\left(c_k = [c_k(E,\nabla)]\right)$

are chern classes of E.

"elliptic operators" $\mathbb{C}$-linear. $D : \Gamma(M,E) \to \Gamma(M,F)$

is Fredholm, $\mathrm{ind}(D) = \dim(\ker D) - \dim(\operatorname{coker} D)$

is determined by characteristic classes of E, F, TM.

The Chern-Weil Construction

The Chern-Weil Construction

The Chern-Weil Construction