Differential Geometry II

Characteristic Classes

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The Hopf Bundle

Recall

$$S^3 := \{\cancel{A}(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2.$$

The Lie group $S^1 = U(1) \overset{= SO(2)}{=} \{z \in \mathbb{C} \mid |z| = 1\}$ is acting on it (from the right) via $z \mapsto zg$. Its quotient is diffeomorphic to

$$S^3/S^1 =: \mathbb{C}P^1 \cong S^2$$

and its quotient map $S^3 \xrightarrow{\pi} S^2$ is a principal S^1-bundle.

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Trivializations are described by

open ball in $\mathbb{C}$

$$(z_1,z_2) \in \overbrace{S^3 \setminus \{(z_1,z_2 \mid z_1 = 0)\}}^{\pi^{-1}(\mathbb{C}P^1\setminus\{[0,1]\})} \mapsto \left(\frac{z_2|z_1|}{z_1}, \frac{z_1}{|z_1|}\right) \in \underbrace{B^2(1)\cancel{\setminus\{(-1,0,)\}}} \times S^1$$

$$(z_1,z_2)\left(\frac{z_1}{|z_1|}\right)^{-1} = \left(|z_1|, \frac{z_2|z_1|}{z_1}\right)$$

and for the second coordinate likewise.

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$$\overset{=\,B^2}{}$$

$$(z_1, z_2) \in S^3 \setminus \{(z_1, z_2 \mid z_1 = 0)\} \overset{\Phi}{\mapsto} \left(\frac{z_2|z_1|}{z_1}, \frac{z_1}{|z_1|}\right) \in B^2(1) \setminus \{(-1, 0,)\} \times$$

$\Phi^{-1}(\zeta, g) = (\sqrt{1-|\zeta|^2}, \zeta)\cdot g$

and for the second coordinate likewise.

B^2 is to be considered with a parametrization
$\varphi : B^2 \to S^2 \setminus \{(-1, 0, 0)\}$. $\varphi(\zeta)$

The Hopf Bundle

$T_p^h S^3 := (T_p \pi^{-1}([p]))^\perp$ defines a connection A of the principal S^1–bundle. $p \in S^3$: • $d\pi_p|_{T_p^h S^3}$ is an isomorphism,

• $d_p r_g (T_p^h S^3) = T_{pg}^h S^3$.

The Hopf Bundle

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Its curvature is described

$$\varphi^* F_A = 2\mathrm{i}dxdy$$

<u>Proof:</u> $T_{(z_1,z_2)}(\pi^{-1}(\pi(z_1,z_2))) = \{ t(iz_1, iz_2) \mid t \in \mathbb{R} \}$

⇝ orthogonal projection to that

$$\tilde{A}_{(z_1,z_2)}(\underbrace{\zeta_1,\zeta_2}_{=\zeta}) = Re\langle \zeta, iz\rangle iz \qquad \|iz\| = 1 \,!$$

$$A_{(z_1,z_2)} = \left(d_e\mu((z_1,z_2),\cdot)\right)^{-1}\tilde{A}_{(z_1,z_2)} \qquad iz \,\tilde{=}\, i \quad i \in \mathfrak{u}(1)$$

$$= Re(\cdot, iz)\cdot i$$

$x_1 + iy_1 = z_1$

$x_2 + iy_2 = z_2$

$$\Rightarrow A_{(z_1,z_2)} = i(-y_1 dx_1 + x_1 dy_1 - y_2 dx_2 + x_2 dy_2)$$

The Hopf Bundle

$$F_A = dA + \frac{1}{i}\underbrace{[A, A]}_{\substack{=0 \text{ since } S^1 \\ \text{is abelean}}} = dA = 2i\,(dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$$

$$(\Phi^{-1})^* F_A = (\Phi^{-1})^* (2i\, dx_2 \wedge dy_2) = 2i\, dx \wedge dy \quad \square \qquad \underline{(x,y) \text{ coord. on } B^2}$$

Therefore $\frac{i}{2\pi}\int_{S^2} F_A = \frac{i}{2\pi}\int_{B^2} \psi^* F_A = -\frac{1}{\pi}\operatorname{vol}(B^2) = -1.$

$$\shortparallel$$

$$\int_{S^2} c_1(S^3 \xrightarrow{\pi} S^2)$$

The Tautological Bundle

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Equivalently, the line bundle $S^3 \times_{\mathrm{id}} \mathbb{C}$ with $\mathrm{id} : S^1 \to S^1 \subset \mathbb{C}^*$ is the tautological line bundle H.

$\mathbb{C}^* = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$

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A connection $\rightsquigarrow$ A^ρ connection on H : $A^\rho = \rho_*(A)$

$F = \rho_*(F_A)$

Thus

$$c_1(S^3 \xrightarrow{\pi} S^2) = c_1(H) = -\omega \in \Omega^2(S^2)$$

with

$$\int_S^{\cancel{2}} \omega = 1.$$

The Axioms and the Chern-Weil Construction

It remains to show that the Chern-Weil forms $c_k(E, \nabla)$ defined by

$$\det(\frac{\mathrm{i}t}{2\pi}F^{\nabla} + \mathrm{id}_E) = \sum_{k=0}^{\infty} c_k(E, \nabla)t^k$$

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(i) Let $f : M \to N$ be a smooth map, $E \xrightarrow{\pi} N$ a complex vector bundle. Then the pull-back connection ∇^f is a connection on $f^*E \xrightarrow{\pi} M$ and $F^{\nabla^f} = f^* F^{\nabla}$.

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and thus

$$c_k(f^*E, \nabla^f) = f^*c_k(E, \nabla). \quad \Rightarrow \quad c_k(f^*E) = f^*c_k(E)$$

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(ii) Let $E_k \stackrel{\pi}{\to} M$, $k = 1, 2$be two complex vector bundles, ∇^k a connection on E_k. Then $\nabla := (\nabla^1, \nabla^2)^T$ defines a connection on $E_1 \oplus E_2$ and

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(iii) We computed $c_1(H) = -\omega$ above. The axiom follows with (iv)
(iv) By definition $c_\ell(E, \nabla)$ are the coefficients of the "characteristic polynomial" of the $k \times k$–matrix (in a local trivialization) $\frac{\mathrm{i}t}{2\pi} F^{\nabla}$ with entries in $\Omega^2(M, \mathbb{C})$, where $k := rk_{\mathbb{C}} E$. Hence, $c_\ell(E, \nabla) = 0$ for all $\ell > k$.

Pontrjagin Classes

Similarly, for a real vector bundle with a connection E, ∇ over a manifold M we may define $\beta_k \in \Omega^{2k}(M; \mathbb{R})$ via

$$\det(tF^{\nabla} + \mathrm{id}_E) = \sum_{k=0}^{\infty} t^k \beta_k(E, \nabla).$$

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Lemma 78: (i) $[\beta_k] = 0$ for k odd.

(ii) $\left(\frac{\mathrm{i}}{2\pi}\right)^k \beta_{2k}(E, \nabla) = c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}, \nabla^{\mathbb{C}})$.

Pontrjagin Classes

Definition 79: Let $E \xrightarrow{\pi} M$ be a real vector bundle. Then

$$p_k(E) := c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4k}_{DR}(M; \mathbb{R})$$

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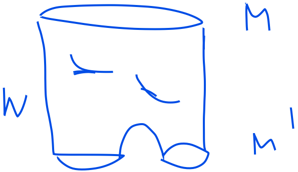

$\dim M = \dim M'$

$\partial W = M \sqcup (-M')$

$\sim$ oriented boundary

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$p_k(M)$ are another theme in the story "Curvature and Topology"!

Hirzebruch's Signature Formula

Let M be a closed oriented $4k$–manifold. Then

$$[\alpha],[\beta] \in H^{2k}_{DR}(M) \mapsto \int_M \alpha \wedge \beta \in \mathbb{R}$$

is a non-degenerate symmetric bilinear form.

· indep't of α & β $\quad \int (\alpha + d\gamma) \wedge \beta$

$$= \int \alpha \wedge \beta + \int d(\gamma \wedge \beta) = \int \alpha \wedge \beta \text{ Stokes}$$

· $\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha = \beta \wedge \alpha \Rightarrow$ symmetric

· represent $[\alpha]$ s.t. $d^* \alpha = \pm * d * \alpha = 0$

$$\int \alpha \wedge * \alpha = \int \|\alpha\|^2 dM > 0$$

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In general, for a closed oriented $4k$–manifold its signature is determined by its Pontrjagin classes. The formula involves the so-called L–genus and was found by Hirzebruch.

Stiefel-Whitney Classes