Differential Geometry II

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Rules for Usage of ZOOM (suject to adjustments)

- switch off your microphones - switch them on while speaking only
- I might ask you to switch off your cameras (if data connection is weak)
- I will regularly pause for questions
- use "raising hand" and comment functions – also if you encounter problems!
- classes will not be recorded
- annotated slides are posted after classes

Organization of Lectures and Tutorials

- information will be given on Moodle and on my homepage (please, see both at the start)
- slides will be posted prior to lectures (if ready)
- no grading of homework **but** prepare them as if
- your solutions and questions will be discussed in tutorials
- office hours: write me an email for an appointment via Skype
- if you cannot skype I can call you under a number you provide
- We will use a whiteboard application for tutorials (certainly via try and error)

Contents of the Class

The new heroes of this class will be **differential forms**. They will appear in each chapter of this class, such as

- Stokes Theorem
- de Rham Cohomology
- connections on vector bundles and their curvature
- symplectic geometry

Let's begin....

Algebraic Preliminiaries

Some of it you might know already...

Definition 1: (1) Let V be a real vector space. A k–linear form is a map

$$\alpha : V^k \longrightarrow \mathbb{R}$$

which is linear in each component:

$$\alpha(v_1, ..., v'_\ell + \lambda v''_\ell, ..., v_k) = \alpha(v_1, ..., v'_\ell, ..., v_k) + \lambda\alpha(v_1, ..., v''_\ell, ..., v_k).$$

for all $v_1, ..., v'_\ell, v''_\ell, ..., v_k \in V$ and $\lambda \in \mathbb{R}$.
One denotes the set of all such k–linear forms by

$$T^k(V^*), T^{0,k}(V), V^* \otimes V^* \otimes ... \otimes V^*, (V^*)^{\otimes k}$$

to name a few.

(2) A k–linear form α is called **symmetric** or **antisymmetric** if for every permutation $\sigma \in S_k$ of $\{1; 2; ...; k\}$ we have

$$\alpha(v_{\sigma(1)}, ..., v_{\sigma(k)}) = \alpha(v_1, ..., v_k)$$

and

$$\alpha(v_{\sigma(1)}, ..., v_{\sigma(k)}) = (-1)^{\sigma} \alpha(v_1, ..., v_k),$$

respectively.
The set of antisymmetric k–linear forms are called **exterior k–forms** (and often later on just k–forms, when there should be no confusion with k–linear forms). It is denoted by $\Lambda^k(V^*)$. For convenience, for $k > \dim V$ or $k < 0$ we set this space to be the trivial vector space.

Examples:

(1) A euclidean metric on a real vector space V is a symmetric bilinear form.
(2) Let $V := \mathbb{R}^n$ be the standard vector space of column vectors. The determinant

$$\det : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n} \to \mathbb{R},$$

where one forms a $n \times n$–matrix by the given n elements of $\mathbb{R}^n$ and takes its determinant, is an exterior n–form of $\mathbb{R}^n$.

Theorem 2: $\Lambda^k(V^*)$ is a real vector space. If $\dim V = n$ then the dimension is given by

$$\dim(\Lambda^k(V^*)) = \binom{n}{k}$$

Proof of Theorem 2:

Pick a basis $\{v_1, \dots, v_n\}$ of V. There are $\binom{n}{k}$ subsets of $\{1; \dots; n\}$ containing exactly k elements. The elements of each subset I will be ordered $I = \{1 \leq i_1 < i_2 < \dots < i_k \leq n\}$ and the subsets will be lexicographically ordered $I_1, I_2, \dots, I_{\binom{n}{k}}$. Hence we define a linear map

$$\Phi : \Lambda^k(V^*) \longrightarrow \mathbb{R}^{\binom{n}{k}}$$

where the ℓ-th component of $\Phi(\alpha)$ is given by

$$\Phi(\alpha)_\ell = \alpha(v_{i_{\ell 1}}, v_{i_{\ell 2}}, \dots, v_{i_{\ell k}})$$

with $I_\ell = \{1 \leq i_{\ell 1} < i_{\ell 2} < \dots < i_{\ell k} \leq n\}$. Claim: Φ is an

isomorphism.

Injectivity of Φ

Assume $\Phi(\alpha) = 0$.
Pick any tupel $w_1, ..., w_k \in V$.

$$w_i = \sum_{j=1}^{n} \lambda_{ij} v_j$$

Then

$$\alpha(w_1, ..., w_k) = \sum_{j_1, j_2, ..., j_k = 1}^{n} \lambda_{1j_1} \lambda_{2j_2} ... \lambda_{kj_k} \underline{\alpha(v_{j_1}, v_{j_2}, ..., v_{j_k})}$$
$$= \sum_{I} c_I \Phi(\alpha)_I = 0$$

where the last sum is over all k–element subsets of $\{1; 2; ...; n\}$.

Hence $\alpha = 0$.

Surjectivity of Φ

Let $I = \{1 \leq i_1 < i_2 < ... < i_k \leq n\}$ be an k–element ordered subset of $\{1, ..., n\}$. We define $\alpha_I \in \Lambda^k(V)$

$$\alpha_I(v_{j_1}, ..., v_{j_k}) = \begin{cases} 0 & \text{if } \{j_1; ...; j_k\} \neq \{i_1; ...; i_k\} \\ (-1)^\sigma & \text{for } \sigma \in S_k \text{ with } j_\ell = i_{\sigma(\ell)}. \end{cases}$$

extending it linearily in each component.
Then $\Phi(\alpha_I) = e_\ell$ with $I_\ell = I$.

Hence Φ is surjective. $\square$

The Wedge–Product

For $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^\ell(V)$ we define $\alpha \wedge \beta \in \Lambda^{k+\ell}(V)$ via

$$(\alpha \wedge \beta)(w_1, \dots w_{k+\ell}) := \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (-1)^\sigma \alpha(w_{\sigma(1)}, \dots w_{\sigma(k)}) \beta(w_{\sigma(k+1)}, \dots w_{\sigma(k+\ell)}).$$

Theorem 3: The wedge–product turns

$$\Lambda^*(V) := \oplus_{k=0}^n \Lambda^k(V).$$

into a *graded commutative algebra* over $\mathbb{R}$.

An **algebra** is a vector space with a (linear) ring structure, **graded algebra** refers to $\alpha \wedge \beta \in \Lambda^{k+\ell}(V)$ for α, β as given above, **graded commutative** means that

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha.$$

The wedge-product is sometimes also called **exterior product**

Proof of Theorem 3:

We will only deal with the associativity. Check the other properties!
For $\alpha \in \Lambda^k(V), \beta \in \Lambda^\ell(V), \gamma \in \Lambda^m(V)$ we define
$\mu(\alpha, \beta, \gamma) \in \Lambda^{k+\ell+m}(V)$ via

$$\mu(\alpha, \beta, \gamma)(w_1, \dots, w_{k+\ell+m}) = \frac{1}{k!\ell!m!} \sum_{\sigma \in S_{k+\ell+m}} (-1)^\sigma \alpha(w_{\sigma(1)}, \dots, w_{\sigma(k)}) \beta(w_{\sigma(k+1)}, \dots, w_{\sigma(k+\ell)}) \times \times \gamma(w_{\sigma(k+\ell+1)}, \dots, w_{\sigma(k+\ell+m)}).$$

Claim: $(\alpha \wedge \beta) \wedge \gamma = \mu(\alpha, \beta, \gamma) = \alpha \wedge (\beta \wedge \gamma)$

Proof of Theorem 3:

$$
\begin{aligned}
&(\alpha \wedge \beta) \wedge \gamma(w_1, \ldots, w_{k+\ell+m}) \\
&\quad = \frac{1}{(k+\ell)!m!} \sum_{\sigma \in S_{k+\ell+m}} (-1)^{\sigma} (\alpha \wedge \beta)\big(w_{\sigma(1)}, \ldots, w_{\sigma(k+\ell)}\big) \times \\
&\qquad \times \gamma\big(w_{\sigma(k+\ell+1)}, \ldots, w_{\sigma(k+\ell+m)}\big) \\
&= \frac{1}{(k+\ell)!m!} \sum_{\sigma \in S_{k+\ell+m}} (-1)^{\sigma} \frac{1}{k!\ell!} \sum_{\tau \in S_{k+\ell}} (-1)^{\tau} \alpha\big(w_{\sigma(\tau(1))}, \ldots w_{\sigma(\tau(k))}\big) \times \\
&\qquad \times \beta\big(w_{\sigma(\tau(k+1))}, \ldots w_{\sigma(\tau(k+\ell))}\big) \times \\
&\qquad \times \gamma\big(w_{\sigma(k+\ell+1)}, \ldots, w_{\sigma(k+\ell+m)}\big) \\
&= \frac{1}{k!\ell!m!} \sum_{\tau \in S_{k+\ell}} \frac{1}{(k+\ell)!} \sum_{\sigma \in S_{k+\ell+m}} (-1)^{\sigma}(-1)^{\tau} \alpha\big(w_{\sigma(\tau(1))}, \ldots w_{\sigma(\tau(k))}\big) \times \\
&\qquad \times \beta\big(w_{\sigma(\tau(k+1))}, \ldots w_{\sigma(\tau(k+\ell))}\big) \times \\
&\qquad \times \gamma\big(w_{\sigma(k+\ell+1)}, \ldots, w_{\sigma(k+\ell+m)}\big)
\end{aligned}
$$

Proof of Theorem 3:

$$= \frac{1}{k!\ell!m!} \sum_{\tau \in S_{k+\ell}} \frac{1}{(k+\ell)!} \sum_{\sigma \in S_{k+\ell+m}} (-1)^{\sigma \circ \tau} \alpha\big(w_{\sigma(\tau(1))}, \ldots w_{\sigma(\tau(k))}\big) \times$$

$$\times \beta\big(w_{\sigma(\tau(k+1))}, \ldots w_{\sigma(\tau(k+\ell))}\big) \times$$

$$\times \gamma\big(w_{\sigma(k+\ell+1)}, \ldots, w_{\sigma(k+\ell+m)}\big)$$

$\lambda = \sigma \circ \tau$

$$= \frac{1}{k!\ell!m!} \sum_{\tau \in S_{k+\ell}} \frac{1}{(k+\ell)!} \sum_{\sigma \in S_{k+\ell+m}} (-1)^{\sigma} \alpha\big(w_{\sigma(1)}, \ldots w_{\sigma(k)}\big) \times$$

$$\times \beta\big(w_{\sigma(k+1)}, \ldots w_{\sigma(k+\ell)}\big) \times$$

$$\times \gamma\big(w_{\sigma(k+\ell+1)}, \ldots, w_{\sigma(k+\ell+m)}\big)$$

$$= \mu(\alpha, \beta, \gamma)(w_1, \ldots, w_{k+\ell+m})$$

The second equality is proved likewise. $\square$

A Basis of $\Lambda^k(V^*)$

Proposition 4: Let $\{v_1, \ldots, v_n\}$ be a basis of the vector spave V and denote by $\{\alpha_1, \ldots, \alpha_n\}$ its dual basis of V^*. Then the elements of the basis $\{\alpha_I \mid I = \{1 \leq i_1 < i_2 < \ldots < i_k\}\}$ of $\Lambda^k(V^*)$ in the proof of Theorem 2 are given by

$$\alpha_{i_1 i_2 \ldots i_k} = \alpha_{i_1} \wedge \alpha_{i_2} \wedge \ldots \wedge \alpha_{i_k}.$$

Proof: It suffices to show that the right hand side evaluated on the k–tupel $(v_{j_1}, v_{j_2}, \ldots, v_{j_k}) \in V^k$ is equal to the left hand side. Now

$$\alpha_{i_1} \wedge \alpha_{i_2} \wedge \ldots \wedge \alpha_{i_k}(v_{j_1}, v_{j_2}, \ldots, v_{j_k})$$
$$= \sum_{\sigma \in S_k} (-1)^\sigma \alpha_{i_1}(v_{j_{\sigma(1)}}) \alpha_{i_2}(v_{j_{\sigma(2)}}) \ldots \alpha_{i_k}(v_{j_{\sigma(k)}})$$

The indices j_k have to be pairwise distinct, since $\alpha_{i_1} \wedge \alpha_{i_2} \wedge \ldots \wedge \alpha_{i_k}$ is antisymmetric by definition.

Proof of Proposition 4:

If the sets $\{i_1, i_2, ..., i_k\} \neq \{j_1, j_2, ..., j_k\}$ at least for one ℓ we have $i_\ell \neq j_{\sigma(\ell)}$, therefore $\alpha_{i_\ell}(v_{j_{\sigma(\ell)}}) = 0$ and hence the whole product vanishes. Since this holds for all σ we have established the vanishing of the right hand side if $\{i_1, i_2, ..., i_k\} \neq \{j_1, j_2, ..., j_k\}$.

Finally, if $\{i_1, i_2, ..., i_k\} = \{j_1, j_2, ..., j_k\}$ there is (exactly one) permutation $\sigma \in S_k$ such that for all ℓ we have $i_\ell = j_{\sigma(\ell)}$ and thus $\alpha_{i_\ell}(v_{j_{\sigma(\ell)}}) = 1$. For all other permutations the corresponding summand hence vanishes and the only summand surviving gives rise to

$$\alpha_{i_1} \wedge \alpha_{i_2} \wedge ... \wedge \alpha_{i_k}(v_{j_1}, v_{j_2}, ..., v_{j_k}) = (-1)^\sigma. \quad \square$$

Pull-back and Interior product

Let $F : V \to W$ be a linear map between real vector spaces V and W. Then the **pull-back** of an exterior k–form $\alpha \in \Lambda^k(W^*)$ is the exterior k–form $F^*\alpha \in \Lambda^k(V^*)$ defined by

$$(F^*\alpha)(v_1, \ldots, v_k) := \alpha(F(v_1), \ldots, F(v_k)).$$

Let $\alpha \in \Lambda^k(V^*)$ be an exterior k–form and $v \in V$ a vector. The **interior product** of v with α is the $(k-1)$–form $v \lrcorner \alpha \in \Lambda^{k-1}(V^*)$ defined by

$$(v \lrcorner \alpha)(v_1, \ldots, v_{k-1}) := \alpha(v, v_1, \ldots, v_{k-1}).$$

Pull-back and Interior Product

Proposition 5: (1) The pull-back $F^* : \Lambda^k(W^*) \to \Lambda^k(V^*)$ is a linear map.
(2) The map $V \times \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$ given by $(v, \alpha) \mapsto v \lrcorner \alpha$ is a bilinear map.
(3) We have the following relations

$$\text{for } v, w \in V, \alpha \in \Lambda^k(V^*) : \quad v \lrcorner (w \lrcorner \alpha) = -w \lrcorner (v \lrcorner \alpha)$$

$$\text{for } v \in V, \alpha \in \Lambda^k(V^*),\ \beta \in \Lambda^\ell(V^*) : \quad v \lrcorner (\alpha \wedge \beta) = (v \lrcorner \alpha) \wedge \beta + (-1)^k \alpha \wedge (v \lrcorner \beta)$$

$$\text{for } F : V \to W \text{ linear},\ v \in V, \alpha \in \Lambda^k(V^*) : \quad v \lrcorner (F^* \alpha) = F^*(F(v) \lrcorner \alpha).$$

Proof: Exercise

Scalar product on $\Lambda^k(V^*)$

Let V be an **oriented, euclidean** vector space. The scalar product induces a scalar product on antisymmetric k–forms: For $\alpha, \beta \in \Lambda^k(V^*)$ we define

$$\langle \alpha, \beta \rangle := \sum_{I=\{1 \leq i_1 < i_2 < \ldots < i_k\}} \alpha(v_{i_1}, \ldots, v_{i_k}) \beta(v_{i_1}, \ldots, v_{i_k})$$

where $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V. This definition does not depend on the orthonormal basis. Moreover, the basis induced by that orthonormal basis, $\{\alpha_I\}_{I=\{1 \leq i_1 < i_2 < \ldots < i_k\}}$ is an orthonormal basis of $\Lambda^k(V^*)$ (Check this!).

The Volume Form

Assume that $\{v_1, ..., v_n\}$ is an n–dimensional, oriented orthonormal basis. The **volume form**, $dV \in \Lambda^n$, of an oriented, euclidean vector space V is defined via

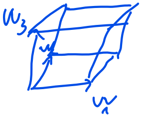
$$dV(w_1, ..., w_n) := \det \begin{pmatrix} \langle w_1, v_1 \rangle & \dots & \langle w_1, v_n \rangle \\ \vdots & \ddots & \vdots \\ \langle w_n, v_1 \rangle & \dots & \langle w_n, v_n \rangle \end{pmatrix}.$$

Lemma 6: (1) The definition of dV is independent of the choice of an oriented orthonormal basis.

(2) It has length one: $\langle dV, dV \rangle = 1$ and $\Lambda^n(V) = \mathbb{R} dV$.

(3) For the dual basis $\{\alpha_1, ..., \alpha_n\}$ of a oriented orthonormal basis as above we have

$$dV = \alpha_1 \wedge ... \wedge \alpha_n.$$



Proof: Exercise

The Hodge-$*$-Operator

Recall that the linear map

$$\alpha \in \Lambda^k(V^*) \mapsto \langle \alpha, . \rangle \in (\Lambda^k(V^*))^*$$

is an isomorhism since $\langle ., . \rangle$ is non-degenerate.

On the other hand for a given $\alpha \in \Lambda^k(V^*)$

$\in \Lambda^n(V^*)$

$$\beta \in \Lambda^{n-k}(V^*) \mapsto \frac{\alpha \wedge \beta}{dV} \in \mathbb{R}$$

defines an element in $(\Lambda^{n-k}(V^*))^*$. Its image under the inverse of the above isomorphism is a $(n-k)$–form, called the **Hodge dual** of α and denoted by $*\alpha \in \Lambda^{n-k}(V^*)$.

The Hodge-$*$-Operator

Lemma 7: (1) The map

$$* : \Lambda^k(V^*) \longrightarrow \Lambda^{n-k}(V^*)$$

is an isometry which is referred to as **Hodge-$*$-operator**.
(2) On k–forms $*^2 = * \circ * = (-1)^{k(n-k)}$.
(3) For $\alpha, \beta \in \Lambda^k(V^*)$ we have

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle dV.$$

Proof of Lemma 7:

Ad (1) $\alpha, \beta \in \Lambda^k(V^*)$

to show $\langle \alpha, \beta \rangle = \langle * \alpha, * \beta \rangle$

by definition: $\langle * \alpha, \gamma \rangle = \dfrac{\alpha \wedge \gamma}{dV} \quad \forall \gamma \in \Lambda^{n-k}(V^*)$

$\Rightarrow \quad \langle * \alpha, \gamma \rangle \, dV = \alpha \wedge \gamma$

Hence $\langle * \alpha, * \beta \rangle \, dV = \alpha \wedge * \beta \quad \ldots ?$

Proof of Lemma 7:

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