Differential Geometry II

Curvature and Global Properties of Riemannian Manifolds

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Jacobi Fields

(M, g) Riemannian manifold

Definition 85: Let $\gamma : I \to M$ be a geodesic. A vector field η along γ is called **Jacobi field** if it satisfies

$$\underbrace{\nabla^{\gamma}_{\frac{\partial}{\partial t}} \nabla^{\gamma}_{\frac{\partial}{\partial t}} \eta}_{= \nabla_{\dot\gamma} \nabla_{\dot\gamma} \eta} + R(\eta, \dot\gamma)\dot\gamma = 0.$$

We had

Proposition 86: (ii) Let $\exp_p : \overset{0_p}{U} \subset T_pM \to M$ be the exponemntial map at $p \in M$, U open starshaped Its differential

$$d_X \exp_p : T_X(T_pM) = T_pM \to T_{\exp_p(X)}M$$

can be described as follows. For $Y \in T_pM$ consider the Jacobi field η along the geodesic $\gamma_X : [0,1] \to M$, with $\gamma_X(0) = p$ $\dot\gamma(0) = X$ with initial conditions

$\exp_p(X) = \gamma_X(1)$

$$\eta(0) = 0, \quad (\nabla_{\dot\gamma}\eta)(0) = Y.$$

Then

$$d_X \exp_p(Y) = \eta(1).$$

Conjugated Points

recall: $d_0 \exp_p = \mathrm{id}_{T_pM} : T_pM \to T_pM \Rightarrow d_X \exp_p$ is isomorphism for $|X| < \varepsilon$ for some $\varepsilon > 0$.

Definition 87: $X \in T_pM$ (or $q = \exp_p(X) \in \operatorname{im} \exp_p$) is called **conjugated to** p if $d_X \exp_p$ is not injective.

$\hat{=}$ q is called conjugated to p along γ_X

Conjugated Points

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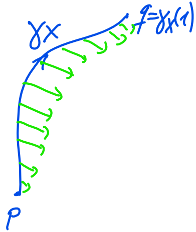
That means, there is a non-trivial Jacobi field Y along $\gamma_X : [0,1] \to M$ with $Y(0) = Y(1) = 0$.



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Proposition 88: Let (M, g) be a Riemannian manifold with non-positive sectional curvature. Then there are no conjugate points along any geodesic.

In particular, for any $p \in M$ the differential of the exponential map $d_X \exp_p : T_pM \to T_{\exp_p(X)}M$ is an isomorphism.

Conjugated Points

Proof: Let $\gamma : [a, b] \to M$ be a geodesic. We define the **index form** of γ on

$$\mathcal{C}(\gamma^* TM) := \{X \in C^0(\gamma^* TM) | X \text{ piecewise smooth, } X(a) = 0 = X(b)\}$$

by

$$I(X, Y) := \int_a^b \left(\underline{g(\nabla_{\dot\gamma} X, \nabla_{\dot\gamma} \cancel{Y})} - g(R(\dot\gamma, X)\dot\gamma, Y)\right) dt$$

Notice that by assumption

$$\star = g(R(X,\dot\gamma)\dot\gamma, X) \quad = \quad -g(R(\dot\gamma, X)\dot\gamma, X) \geq 0$$

Hence $I(X, X) > 0$ for any $X \neq 0$.

If there were $a \leq t_0 < t_1 \leq b$ and a non-trivial Jacobi field X along $\gamma|_{[t_0,t_1]}$ with $X(t_0) = X(t_1) = 0$ then we would find (continuing X by zero outside $[t_0, t_1]$)

$$\star = -K(\mathrm{span}(\dot\gamma, X)) \cdot \underbrace{\|X \wedge \dot\gamma\|^2}_{\geq 0}$$

check the sign!

Conjugated Points

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$$\begin{aligned} 0 < I(X, X) &= \int_{t_0}^{t_1} \left(g(\nabla_{\dot\gamma} X, \nabla_{\dot\gamma} X) - g(R(\dot\gamma, X)\dot\gamma, X)\right) dt \\ &= \int_{t_0}^{t_1} \left(-g(\nabla_{\dot\gamma}\nabla_{\dot\gamma} X + R(\dot\gamma, X)\dot\gamma, X)\right) dt = 0 \quad \square \end{aligned}$$

Hadamard Manifolds

Manifolds with non-positive sectional curvature are called **Hadamard manifolds**. Important examples are flat manifolds ($K = 0$) and hyperbolic space ($K = -1$).

$$\mathbb{H}^n = \{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0 \} \qquad g_{\mathbb{H}} = \frac{1}{x_n^2} g_{eucl}.$$

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Theorem 89: Let (M, g) be a complete connected Riemannian manifold.
(1) Assume the sectional curvature is non-positive. Then for any point the exponential map $\exp_p : T_pM \to M$ is a covering of M. In particular, it is isomorphic to the universal covering of M and $\pi_k(M) = 0$ for any $k \geq 2$, i.e. any continuous map $u : S^k \to M$ is homotopic to a constant map. If M was simply connected, $\exp_p$ is diffeomorphism for any p.

cov: $\forall p \in M \ \exists U \subset M, p \in U$ s.t. $\exp_p^{-1}(U) = \coprod_{i \in I} V_i$
$\exp_p|_{V_i} : V_i \to U$ is a diffeom.

$u :$
$$\begin{tikzcd} & T_pM \arrow[d, "\exp_p"] \\ S^k \arrow[ur, dashed] \arrow[r] & M \end{tikzcd}$$
$k \geq 2$

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(2) Assume in addition that K is constant, $K \leq 0$. Then $(T_pM, \exp_p^* g)$ is isometric to the euclidean space if $K = 0$ or $(\mathbb{H}^n, \lambda^2 g_{\mathbb{H}})$ for an appropriate λ if $K < 0$.

Hadamard Manifolds

Comment on Proof: (1) Prop 88 $\Rightarrow$ $\forall\, X \in T_pM$ $\exists\, V_X \subset T_pM$ open, $X \in V_X$

& $\exp_p|_{V_X} : V_X \to \exp_p(V) \subset M$ is a diffeomorphism [onto] an open subset of M

(inverse function theorem).

Let $\{ V_X \mid X \in \pi^{-1}(q) \}$.

claim: can choose V_X s.t.

- $V_{X_1} \cap V_{X_2} = \emptyset$ for $X_1 \neq X_2$
- $\exp_p(V_{X_1}) = \exp_p(V_{X_2})$

% $\exp_p$ is surjective

% $q \in M$ $\exists$ geodesic $\|\dot\gamma\| = 1$, $\gamma : [0, d(p,q)] \to M$

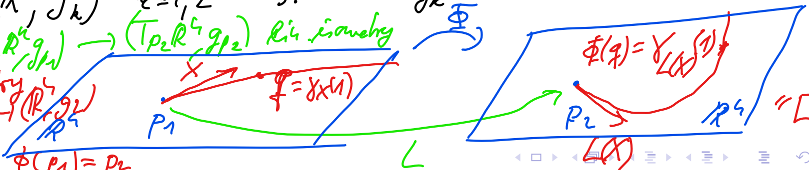
% $\gamma(0) = p$, $\gamma(d(p,q)) = q$

% $q = \exp_p(X)$, $X = d(p,q)\dot\gamma(0)$

(2) $(\mathbb{R}^n, g_k)$ $k = 1,2$ s.t. K_{g_k} are constant & ≤ 0 & equal.

Fix $L : (T_{p_1}\mathbb{R}^n, g_{p_1}) \to (T_{p_2}\mathbb{R}^n, g_{p_2})$ lin. isometry

$\exists!$ isometry $\Phi : (\mathbb{R}^n, g_1) \to (\mathbb{R}^n, g_2)$, $d_{p_1}\Phi = L$, $\Phi(p_1) = p_2$



"□"

Positive Curvature

If $K > 0$ on some tangent plane at a point of the geodesic γ containing $\dot{\gamma}$. The index form can become indefinite or degenerate.

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[Bonnet]

Theorem 90: Assume that (M, g) is a complete connected Riemannian manifold with sectional curvature

$$K \geq \frac{1}{R^2}. > 0$$

Then the diameter of M is bounded by

$$\mathrm{diam}(M, g) \leq \pi R.$$

$\Leftarrow$ $\exp_p|_{B_{g_p}(\pi R)}$ is surjective

In particular, M is compact and its fundamental group is finite.
Remark: There is a similar result by Myers replacing the condition on the sectional curvature by on on the Ricci curvature.

Positive Curvature

Rough idea: • Jacobi equations are cases of Sturm-Liouville-equations

$X: I \to \mathbb{R}^n \qquad A: I \to M(n,\mathbb{R}) \quad A^T = A$

$$X''(t) + A(t)X(t) = 0$$

$\exists?\ X(t_0) = X(t_1) = 0$ for some $t_0 < t_1 \in I$, $X \not\equiv 0$.

Example: $A = k^2 E_n \qquad X(t) = a \cdot \cos\left(\frac{t}{k}\right) + b \sin\left(\frac{t}{k}\right)$

$a, b \in \mathbb{R}^n$.

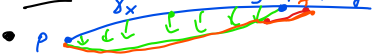
$I = \mathbb{R}$, $t_0 = 0 \qquad |t_1 - t_0| = \pi k N \qquad N \in \mathbb{N}$

if nontrivial X exist.

Assume A positive definite & smallest eigenvalue $\geq k^2 > 0$

claim: $\exists$ nontrivial X for $t_0 = 0$, some $t_1 > 0$: $t_1 < \pi k$

$\exists\ q\ q'$ conjugated pair: $d(p,q) < \pi R$



Symplectic Manifolds

Definition 90: Let M be a smooth manifold. A **symplectic structure** of **symplectic form** on M is a closed, non-degenerate 2-form $\omega \in \Omega^2(M)$, i.e. $d\omega = 0$ and for all $p \in M$

$$X \in T_pM \mapsto X \lrcorner \omega \in T_p^*M$$

is an isomorphism.
Lemma 91: (1) The non-degeneracy implies that $\dim M = 2n$ is even.
(2) It is equivalent to

$$\omega^n = \omega \wedge \ldots \wedge \omega \neq 0$$

is a volume form. In particular, M has to be oriented.
(3) If M is a closed manifold, then

$$b_2(M) := \dim H^2_{DR}(M) \geq 1.$$

Symplectic Manifolds

Examples