Differential Geometry II

Curvature and Global Properties of Riemannian Manifolds

Klaus Mohnke

July 7, 2020

Jacobi Fields

Definition 85: Let $\gamma : I \to M$ be a geodesic. A vector field ξ along γ is called **Jacobi field** if it satisfies

$$\nabla^{\Gamma}_{\frac{\partial}{\partial t}} \nabla^{\Gamma}_{\frac{\partial}{\partial t}} \eta + R(\eta, \dot{\gamma})\dot{\gamma} = 0.$$

We had

Proposition 86: (i) Let $\gamma : I \to M$ be a geodesic and ξ a Jacobi field along γ.

Then

$$\xi(t) = \xi_0(t) + (a + bt)\dot{\gamma}(t)$$

for a Jacobi field ξ_0 along γ with $g(\xi_0(t), \dot{\gamma}(t)) \equiv 0$.

(ii) Let $\exp_p : U \subset T_pM \to M$ be the exponemntial map at $p \in M$, U open starshaped Its differential

$$d_X \exp_p : T_X(T_pM) = T_pM \to T_{\exp_p(X)}M$$

can be described as follows. For $Y \in T_p$ consider the Jacobi field η along the geodesic $\gamma_X : [0,1] \to M$, with $\gamma_X(0) = p$ $\dot{\gamma}(0) = X$ with initial conditions

$d_X \exp_p(Y) = \eta(1)$

$$\eta(0) = 0, \quad \nabla_{\dot{\gamma}}\eta(0) = Y.$$

Conjugated Points

Definition 87: $X \in T_pM$ is called **conjugated to** p if $d_X \exp_p$ is not injective. Accordingly, a point $q = \gamma(a)$ is conjugated to a point $p = \gamma(b)$ if $q = \exp_p(X)$, $\gamma = \gamma_X$ up to translation and X is conjugated to p.

Conjugated Points

Definition 87: $X \in T_pM$ is called **conjugated to** p if $d_X \exp_p$ is not injective. Accordingly, a point $q = \gamma(a)$ is conjugated to a point $p = \gamma(b)$ if $q = \exp_p(X)$, $\gamma = \gamma_X$ up to translation and X is conjugated to p.

That means, there is a non-trivial Jacobi field Y along $\gamma_X : [0,1] \to M$ with $Y(0) = Y(1) = 0$. Hence, if q is conjugated to p along γ then p is conjugated to q along $\bar{\gamma}$ - the geodesic γ with a opposite parametrization.
In particular, if $\gamma(a)$ and $\gamma(b)$ are not conjugated along the geodesic γ than a Jacobi field along γ is uniquely determined by its values at a and b.

Conjugated Points

Definition 87: $X \in T_pM$ is called **conjugated to** p if $d_X \exp_p$ is not injective. Accordingly, a point $q = \gamma(a)$ is conjugated to a point $p = \gamma(b)$ if $q = \exp_p(X)$, $\gamma = \gamma_X$ up to translation and X is conjugated to p.

That means, there is a non-trivial Jacobi field Y along $\gamma_X : [0, 1] \to M$ with $Y(0) = Y(1) = 0$. Hence, if q is conjugated to p along γ then p is conjugated to q along $\bar{\gamma}$ - the geodesic γ with a opposite parametrization.
In particular, if $\gamma(a)$ and $\gamma(b)$ are not conjugated along the geodesic γ than a Jacobi field along γ is uniquely determined by its values at a and b.

Proposition 88: Let (M, g) be a Riemannian manifold with non-positive sectional curvature. Then there are no conjugate points along any geodesic.

Conjugated Points

Definition 87: $X \in T_pM$ is called **conjugated to** p if $d_X \exp_p$ is not injective. Accordingly, a point $q = \gamma(a)$ is conjugated to a point $p = \gamma(b)$ if $q = \exp_p(X)$, $\gamma = \gamma_X$ up to translation and X is conjugated to p.

That means, there is a non-trivial Jacobi field Y along $\gamma_X : [0,1] \to M$ with $Y(0) = Y(1) = 0$. Hence, if q is conjugated to p along γ then p is conjugated to q along $\bar{\gamma}$ - the geodesic γ with a opposite parametrization.
In particular, if $\gamma(a)$ and $\gamma(b)$ are not conjugated along the geodesic γ than a Jacobi field along γ is uniquely determined by its values at a and b.

Proposition 88: Let (M, g) be a Riemannian manifold with non-positive sectional curvature. Then there are no conjugate points along any geodesic.

In particular, for any $p \in M$ the differential of the exponential map $d_X \exp_p : T_pM \to T_{\exp_p}M$ is an isomorphism.

Conjugated Points

Proof: Let $\gamma : [a, b] \to M$ be a geodesic. We define the **index form** of γ on

$$\mathcal{C}(\gamma^* TM) := \{X \in C^0(\gamma^* TM) | X \text{ piecewise smooth, } X(a) = 0 = X(b)\}$$

by

$$I(X, Y) := \int_a^b \left(g(\nabla_{\dot\gamma} X, \nabla_{\dot\gamma} X) - g(R(X, \dot\gamma)\dot\gamma, Y) \right) dt$$

Notice that by assumption

$$g(R(X, \dot\gamma)\dot\gamma, X) \leq 0$$

Hence $I(X, X) > 0$ for any X with $\nabla_{\dot\gamma} \neq 0$.
If there were $a \leq t_0 < t_1 \leq b$ and a non-trivial Jacobi field X along $\gamma|_{[t_0,t_1]}$ with $X(t_0) = X(t_1) = 0$ then we would find (continuing X by zero outside $[t_0, t_1]$)

Conjugated Points

Proof: Let $\gamma : [a, b] \to M$ be a geodesic. We define the **index form** of γ on

$\mathcal{C}(\gamma^* TM) := \{X \in C^0(\gamma^* TM) | X \text{ piecewise smooth}, X(a) = 0 = X(b)\}$

by

$$I(X, Y) := \int_a^b \left(g(\nabla_{\dot\gamma} X, \nabla_{\dot\gamma} X) - g(R(X, \dot\gamma)\dot\gamma, Y)\right) dt$$

Notice that by assumption

$$g(R(X, \dot\gamma)\dot\gamma, X) \leq 0$$

Hence $I(X, X) > 0$ for any X with $\nabla_{\dot\gamma} \neq 0$.
If there were $a \leq t_0 < t_1 \leq b$ and a non-trivial Jacobi field X along $\gamma|_{[t_0,t_1]}$ with $X(t_0) = X(t_1) = 0$ then we would find (continuing X by zero outside $[t_0, t_1]$)

$$0 < I(X, X) = \int_{t_0}^{t_1} \left(g(\nabla_{\dot\gamma} X, \nabla_{\dot\gamma} X) + g(R(X, \dot\gamma)\dot\gamma, X)\right) dt$$

$$= -\int_{t_0}^{t_1} \left(g(\nabla_{\dot\gamma}\nabla_{\dot\gamma} X + R(X, \dot\gamma)\dot\gamma, X)\right) dt = 0 \quad \square$$

Hadamard Manifolds

Manifolds with non-positive sectional curvature are called **Hadamard manifolds**. Important examples are flat manifolds ($K = 0$) and hyperbolic space ($K = -1$).

Hadamard Manifolds

Manifolds with non-positive sectional curvatire are called **Hadamard manifolds**. Important examples are flat manifolds ($K = 0$) and hyperbolic space ($K = -1$).

Theorem 89: Let (M, g) be a complete Riemannian manifold. (1) Assume the sectional curvature is non-positive. Then for any point the exponential map $\exp_p : T_pM \to M$ is a covering of M. In particular, it is isomorphic to the universal covering of M and $\pi_k(M) = 0$ for any $k \geq 2$, i.e. any continuous map $u : S^k \to M$ is homotopic to a constant map. If M was simply connected, $\exp_p$ is diffeomorphism for any p.

Hadamard Manifolds

Manifolds with non-positive sectional curvatire are called **Hadamard manifolds**. Important examples are flat manifolds ($K = 0$) and hyperbolic space ($K = -1$).

Theorem 89: Let (M, g) be a complete Riemannian manifold.
(1) Assume the sectional curvature is non-positive. Then for any point the exponential map $\exp_p : T_pM \to M$ is a covering of M. In particular, it is isomorphic to the universal covering of M and $\pi_k(M) = 0$ for any $k \geq 2$, i.e. any continuous map $u : S^k \to M$ is homotopic to a constant map. If M was simply connected, $\exp_p$ is diffeomorphism for any p.

(2) Assume in addition that K is constant, $K \leq 0$. Then $(T_pM, \exp_p^* g)$ is isometric to the euclidean space if $K = 0$ or $(\mathbb{H}^n, \lambda^2 g_{\mathbb{H}})$ for an appropriate λ if $K < 0$.

Hadamard Manifolds

Proof: (1) On T_pM define $h = \exp_p^* g$. $\exp_p$ non-degenerate since ... is an isomorphism.

Geodesics w.r.t. h starting at $0 \in T_pM$ are the lines through 0, for ...

$\gamma : [0, L] \to T_pM$, $\gamma(t) = tX$: $\ell_h(\gamma) = \ell_g(\exp_p \circ \gamma) = \ell_g(\gamma_X|[0,L]) = L \cdot \|X\|_{g_p}$

$\Rightarrow (T_pM, h)$ complete! ... lines of T_pM through 0 &

(2) $\Phi : (N, h) \to (M, g)$, (N, h) complete, Φ local isometry, M connected

Claim: (T_pM, h) a covering: $\forall p \in M$ $\exists$ open nbhd $U \subset M$ of p s.t.

(2) $\Phi^{-1}(U) = \coprod_{q \in \Phi^{-1}(p)} V_q$, $\Phi|_{V_q} : V_q \to U$ a diffeo, $V_q \cap V_{q'} = \emptyset$ $\forall q \neq q'$.

Exercise: see Cheeger/Ebin

In case: ( ) $\to (M, \ )$

... even ...)

Hadamard Manifolds

we know: $p \in M$ & $q \in \Phi^{-1}(p)$ $\exists$ $U_q \subset M$ open nbhd of p in M and $V_q \subset \mathcal{U}$ of q open nbhd. s.t. $\Phi|V_q : V_q \simeq U_q$.

<u>aim:</u> $\exists$ $W \subset M$ open nbhd. of p s.t. $\Phi^{-1}(W) = \bigsqcup_{q \in \Phi^{-1}(p)} W_q$ s.t. $\Phi|W_q : W_q \to W$

Positive Curvature

If $K > 0$ on some tangent plane at a point of the geodesic γ containing $\dot{\gamma}$ the index form can become indefinite or degenerate.

Positive Curvature

If $K > 0$ on some tangent plane at a point of the geodesic γ containing $\dot{\gamma}$ the index form can become indefinite or degenerate.

Theorem 90: Assume that (M, g) is a complete Riemannian manifold with one of the following bounds on its curvature: (i) For the sectional curvature we have

$$K \geq \frac{1}{R^2}, \text{ or}$$

(ii) the Ricci curvature satisfies

$$Ric(v, v) \geq \frac{n-1}{R^2}$$

for all unit tangent vectors $v \in T_pM$.

Positive Curvature

If $K > 0$ on some tangent plane at a point of the geodesic γ containing $\dot{\gamma}$ the index form can become indefinite or degenerate.

Theorem 90: Assume that (M, g) is a complete Riemannian manifold with one of the following bounds on its curvature: (i) For the sectional curvature we have

$$K \geq \frac{1}{R^2}, \text{ or}$$

(ii) the Ricci curvature satisfies

$$Ric(v, v) \geq \frac{n-1}{R^2}$$

for all unit tangent vectors $v \in T_pM$.

Recall: $p \in M$, $\{e_i\}$ ONB of (T_pM, g_p), $v, w \in T_pM$

$$Ric(v,w) := \sum_{i=1}^{n} g(R(e_i, v)w, e_i)$$

Then the diameter of M is bounded by

$$\text{diam}(M, g) \leq \pi R.$$

In particular, M is compact and its fundamental group is finite.

Index Lemma

We need the following Lemma on the index:

Lemma 91: Assume for a geodesic $\gamma : [a, b] \to M$ that there is no $t \in [a, b]$ such that $\gamma(t)$ is conjugated to $\gamma(a)$ along γ. Let X a piecewise smooth vector field along γ and ξ be the unique Jacobi field such that $\xi(a) = X(a) = 0$ and $\xi(b) = X(b)$. Then

$$I(\xi, \xi) \leq I(X, X)$$

and equality holds if and only if $X = \xi$. ξ Jacobi field

Proof: $(X-\xi)(b) = 0$. & piecewise smooth. $\Rightarrow I(X-\xi, \xi) = 0$

$\Rightarrow 0 \leq I(X-\xi, X-\xi) = I(X-\xi, X+\xi) = I(X,X) - I(\xi,\xi)$

Suff. to show: For any piecewise smooth vectorfield Y along γ, with $Y(a) = Y(b) = 0$ we have $I(Y,Y) \geq 0$ & "=" $\Leftrightarrow Y = 0$.

There is an orthonormal frame $\{E_i\}$ along γ, $\nabla_{\dot\gamma} E_i \equiv 0$.
$Y = \sum y_i \cdot E_i$ & the index form & Jacobi equation translate into

Index Lemma

$y : [a,b] \longrightarrow \mathbb{R}^n$ $\qquad R(\sum y_i E_i, \dot\gamma)\dot\gamma = \sum A_{ij} y_j E_i$

$A : [a,b] \to \mathrm{Symm}(n, \mathbb{R})$

$-y''(t) + A(t)y(t) = 0 \quad (*)$

$$I(x,y) = \int_a^b (\langle x', y'\rangle + \langle A(t)x, y\rangle)\,dt$$

<u>Ass</u>: If y satisfies $(*)$ & $y(a) = y(t) = 0$ for some $t \in (a,b]$ then $y \equiv 0$.

<u>Claim</u>: for all x with $x(a) = x(b) = 0$ $I(x,x) \geq 0$, x piecewise smooth & "$=$" $\Longleftrightarrow$ $x \equiv 0$.

<u>Proof</u>: $x(a) = x(b) = 0$. Define $\{u_i\}_{i=1}^n$ u_i satisfies $(*)$

$u_i(a) = 0$, $u_i(b) = e_i$.

<u>Claim</u>: $x(t) = \sum f_i u_i$ $\quad$ f_i piecewise smooth on $(a,b]$, $f_i'(a)$ exists. $f_i(b) = 0$.

$u_i(a) = 0 \longrightarrow \dfrac{u_i}{t-a} =: v_i$ is differentiable (also at a)

$\Rightarrow x(t) = \sum g_i(t)\, v_i(t) = \sum g_i(t)(t-a)\, u_i(t)$

Index Lemma

also $\left(\langle u_i', u_j\rangle - \langle u_i, u_j'\rangle\right)' = \langle A(t)u_i, u_j\rangle - \langle u_i, A(t)u_j\rangle = 0$

constant & $= 0$ at $t = a$ $\Rightarrow \equiv 0$ $\leftarrow$ $(*)$

$$x'(t) = \sum (f_i' u_i + f_i u_i')$$

$$I(x,x) = \sum \int f_i' f_j' \langle u_i, u_j\rangle + f_i' f_j \langle u_i, u_j'\rangle + f_i f_j' \langle u_i', u_j\rangle + f_i f_j \langle u_i', u_j'\rangle + f_i f_j \langle A u_i, u_j\rangle \, dt = (**)$$

$$\sum \int f_i f_j \langle u_i', u_j'\rangle \, dt = -\sum \int f_i' f_j \langle u_i', u_j\rangle + f_i f_j' \langle u_i', u_j\rangle + f_i f_j \langle u_i'', u_j\rangle$$

$$= -\sum_i \int f_i' f_j \langle u_i, u_j'\rangle + f_i f_j' \langle u_i', u_j\rangle + f_i f_j \langle A u_i, u_j\rangle \quad (*)$$

$$(**) = \sum_{i,j} \int f_i' f_j' \langle u_i, u_j\rangle = \int \|f\|_g^2 \geq 0 \qquad g_{ij} = \langle u_i, u_j\rangle$$

symmetric & positive definite

$= 0 \Leftrightarrow f' \equiv 0$ & since $f(b) = 0 \Rightarrow f \equiv 0$. $\square$

Index Lemma

Corollary: Let $\gamma : [0, T) \to M$ be a geodesic and $\gamma(t_0)$ conjugate to $\gamma(0)$ along γ. Then $\gamma|_{[0,t]}$ is not minimal for any $t > t_0$.

Index Lemma

Corollary: Let $\gamma : [0, T) \to M$ be a geodesic and $\gamma(t_0)$ conjugate to $\gamma(0)$ along γ. Then $\gamma|_{[0,t]}$ is not minimal for any $t > t_0$.

Proof: W.l.o.g. there are no conjugate points in $\gamma|_{[0,t_0)}$.

Index Lemma

Corollary: Let $\gamma : [0, T) \to M$ be a geodesic and $\gamma(t_0)$ conjugate to $\gamma(0)$ along γ. Then $\gamma|_{[0,t]}$ is not minimal for any $t > t_0$.

Proof: W.l.o.g. there are no conjugate points in $\gamma|_{[0,t_0)}$.
Let $J \neq 0$ be a Jacobi field along $\gamma|_{[0,t_0]}$ such that
$J(0) = J(t_0) = 0$ and extend it to vector field X along γ by zero.
$I(X, X) = 0$ on $[0, t]$.

Index Lemma

Corollary: Let $\gamma : [0, T) \to M$ be a geodesic and $\gamma(t_0)$ conjugate to $\gamma(0)$ along γ. Then $\gamma|_{[0,t]}$ is not minimal for any $t > t_0$.

Proof: W.l.o.g. there are no conjugate points in $\gamma|_{[0,t_0)}$.

Let $J \neq 0$ be a Jacobi field along $\gamma|_{[0,t_0]}$ such that $J(0) = J(t_0) = 0$ and extend it to vector field X along γ by zero. $I(X, X) = 0$ on $[0, t]$.

$\delta > 0$ small, so that no conjugate points on $\gamma|_{[t_0-\delta,t_0+\delta]}$. Let Z be the Jacobi field along $\gamma|_{[t_0-\delta,t_0+\delta]}$ with $Z(t_0 - \delta) = J(t_0 - \delta)$, $Z(t_0 + \delta) = 0$.

Index Lemma

Corollary: Let $\gamma : [0, T) \to M$ be a geodesic and $\gamma(t_0)$ conjugate to $\gamma(0)$ along γ. Then $\gamma|_{[0,t]}$ is not minimal for any $t > t_0$.

Proof: W.l.o.g. there are no conjugate points in $\gamma|_{[0,t_0)}$.

Let $J \neq 0$ be a Jacobi field along $\gamma|_{[0,t_0]}$ such that $J(0) = J(t_0) = 0$ and extend it to vector field X along γ by zero. $I(X, X) = 0$ on $[0, t]$.

$\delta > 0$ small, so that no conjugate points on $\gamma|_{[t_0-\delta,t_0+\delta]}$. Let Z be the Jacobi field along $\gamma|_{[t_0-\delta,t_0+\delta]}$ with $Z(t_0 - \delta) = J(t_0 - \delta)$, $Z(t_0 + \delta) = 0$. Define

$$Y(t) := \begin{cases} J(t) & \text{for } t \in [0, t_0 - \delta] \\ Z(t) & \text{for } t \in [t_0 - \delta, t_0 + \delta] \\ 0 & \text{for } t > t_0 + \delta \end{cases}$$

Index Lemma

Corollary: Let $\gamma : [0, T) \to M$ be a geodesic and $\gamma(t_0)$ conjugate to $\gamma(0)$ along γ. Then $\gamma|_{[0,t]}$ is not minimal for any $t > t_0$.

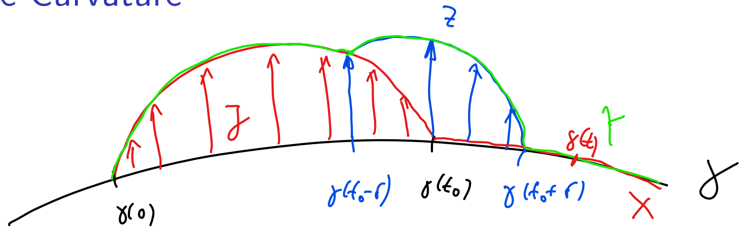
Proof: W.l.o.g. there are no conjugate points in $\gamma|_{[0,t_0)}$.

Let $J \neq 0$ be a Jacobi field along $\gamma|_{[0,t_0]}$ such that $J(0) = J(t_0) = 0$ and extend it to vector field X along γ by zero. $I(X, X) = 0$ on $[0, t]$.

$\delta > 0$ small, so that no conjugate points on $\gamma|_{[t_0-\delta,t_0+\delta]}$. Let Z be the Jacobi field along $\gamma|_{[t_0-\delta,t_0+\delta]}$ with $Z(t_0 - \delta) = J(t_0 - \delta)$, $Z(t_0 + \delta) = 0$. Define

$$Y(t) := \begin{cases} J(t) & \text{for } t \in [0, t_0 - \delta] \\ Z(t) & \text{for } t \in [t_0 - \delta, t_0 + \delta] \\ 0 & \text{for } t > t_0 + \delta \end{cases}$$

Positive Curvature



X is not smooth at $t_0 \Rightarrow X$ not Jacobi

Z is Jacobi & $Z(t_0-\delta) = X(t_0-\delta)$, $Z(t_0+\delta) = X(t_0+\delta) = 0$

Index lemma $\Rightarrow$ $I(Z,Z) < I(X,X)$ on $[t_0-\delta, t_0+\delta]$

$+$ $I(Z,Z) = I(X,X)$ on $(0, t_0-\delta]$ $\quad X = Z$

$I(Z,Z) < I(X,X) = 0$ on $(0, t_0+\delta]$

Positive Curvature

X is not a Jacobi field along $\gamma|_{[t_0-\delta, t_0+\delta]}$. Hence on $[t_0 - \delta, t_0 + \delta]$

$$I(Z, Z) < I(X, X).$$

Positive Curvature

X is not a Jacobi field along $\gamma|_{[t_0-\delta, t_0+\delta]}$. Hence on $[t_0-\delta, t_0+\delta]$

$$I(Z,Z) < I(X,X). = 0$$

Since $Y = X$ outside this interval we have $I(Y,Y) < I(X,X) = 0$ on $[0,t]$. $\square$

$\gamma_T(a) = \gamma(a)$
$\gamma_T(t) = \gamma(t)$

use Y to deform $\gamma \rightsquigarrow \gamma_T$ piecewise smooth curves:

$$\mathcal{E}(\gamma_T) = \mathcal{E}(\gamma) + T^2 \underbrace{I(Y,Y)}_{<0} + O(T^3) < \mathcal{E}(\gamma)$$

$\Rightarrow \ell(\gamma_T) < \ell(\gamma)$ for $T > 0$ small

no linear term in Taylor series since $\gamma = \gamma_0$ is a geodesic!

Proof of Bonnet's Theorem

Consider a geodesic $\gamma : [0, L] \to M$. Let X be vector field along γ, $X \perp \dot{\gamma}$ and $\nabla_{\dot{\gamma}} X = 0$. Define $Y(t) := \sin(\pi t/L)$. Then

Proof of Bonnet's Theorem

Consider a geodesic $\gamma : [0, L] \to M$. Let X be vector field along γ, $X \perp \dot{\gamma}$ and $\nabla_{\dot{\gamma}} X = 0$. Define $Y(t) := \sin(\pi t/L)$. Then

$$\begin{aligned} I(Y, Y) &= -\int_0^L g(Y, \nabla_{\dot{\gamma}}^2 Y + R(Y, \dot{\gamma})\dot{\gamma})dt \\ &= \int_0^L (\sin(\pi/L))^2(\pi^2/L^2 - g(R(X, \dot{\gamma})\dot{\gamma}, X))dt \end{aligned}$$

Proof of Bonnet's Theorem

Consider a geodesic $\gamma : [0, L] \to M$. Let X be vector field along γ, $X \perp \dot{\gamma}$ and $\nabla_{\dot{\gamma}} X = 0$. Define $Y(t) := \sin(\pi t/L)$. Then

$$\begin{aligned} I(Y, Y) &= -\int_0^L g(Y, \nabla^2_{\dot{\gamma}} Y + R(Y, \dot{\gamma})\dot{\gamma})dt \\ &= \int_0^L (\sin(\pi/L))^2(\pi^2/L^2 - g(R(X, \dot{\gamma})\dot{\gamma}, X))dt \end{aligned}$$

Let $L \geq \pi R$. Then $K \geq 1/R^2$ implies hat $I(Y, Y) \leq 0$.

Proof of Bonnet's Theorem

Consider a geodesic $\gamma : [0, L] \to M$. Let X be vector field along γ, $X \perp \dot{\gamma}$ and $\nabla_{\dot{\gamma}} X = 0$. Define $Y(t) := \sin(\pi t/L)$. Then

$$\begin{aligned} I(Y, Y) &= -\int_0^L g(Y, \nabla_{\dot{\gamma}}^2 Y + R(Y, \dot{\gamma})\dot{\gamma}) dt \\ &= \int_0^L (\sin(\pi/L))^2 (\pi^2/L^2 - g(R(X, \dot{\gamma})\dot{\gamma}, X)) dt \end{aligned}$$

Let $L \geq \pi R$. Then $K \geq 1/R^2$ implies hat $I(Y, Y) \leq 0$.

Assume γ has no conjugate points. Then the unique Jacobi field J with $Z(0) = Y(0) = 0$ and $Z(L) = Y(L) = 0$ (i.e. $J \equiv 0$) would satisfy

$$0 = I(J, J) < I(Y, Y) \leq 0$$

giving a contradiction.

Proof of Bonnet's Theorem

Consider a geodesic $\gamma : [0, L] \to M$. Let X be vector field along γ, $X \perp \dot{\gamma}$ and $\nabla_{\dot{\gamma}} X = 0$. Define $Y(t) := \sin(\pi t/L)$. Then

$$\begin{aligned} I(Y, Y) &= -\int_0^L g(Y, \nabla^2_{\dot{\gamma}} Y + R(Y, \dot{\gamma})\dot{\gamma})dt \\ &= \int_0^L (\sin(\pi/L))^2 (\pi^2/L^2 - g(R(X, \dot{\gamma})\dot{\gamma}, X))dt \end{aligned}$$

Let $L \geq \pi R$. Then $K \geq 1/R^2$ implies hat $I(Y, Y) \leq 0$.

Assume γ has no conjugate points. Then the unique Jacobi field J with $Z(0) = Y(0) = 0$ and $Z(L) = Y(L) = 0$ (i.e. $J \equiv 0$) would satisfy

$$0 = I(J, J) < I(Y, Y) \leq 0$$

giving a contradiction.

Hence γ has conjugate points and is not minimal. □

Proof of Bonnet's Theorem

Symplectic Manifolds

Definition 90: Let M be a smooth manifold. A **symplectic structure** of **symplectic form** on M is a closed, non-degenerate 2-form $\omega \in \Omega^2(M)$, i.e. $d\omega = 0$ and for all $p \in M$

$$X \in T_pM \mapsto X \lrcorner \omega \in T_p^*M$$

is an isomorphism.

Lemma 91: (1) The non-degeneracy implies that $\dim M = 2n$ is even.

(2) It is equivalent to

$$\omega^n = \omega \wedge \ldots \wedge \omega \neq 0$$

is a volume form. In particular, M has to be oriented.

(3) If M is a closed manifold, then

$$b_2(M) := \dim H^2_{DR}(M) \geq 1.$$

Symplectic Manifolds

Examples