Differential Geometry II

Symplectic Manifolds

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July 9, 2020

Bonnet's Theorem

Theorem 90: Assume that (M, g) is a complete connected Riemannian manifold for which the sectional curvature satisfies

$$K \geq \frac{1}{R^2}$$

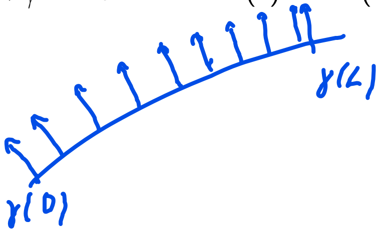
Then the diameter of M is bounded by

$$\operatorname{diam}(M, g) \leq \pi R.$$

In particular, M is compact and its fundamental group is finite.

Proof of Bonnet's Theorem

Consider a geodesic $\gamma : [0, L] \to M$. Let X be vector field along γ, $X \perp \dot{\gamma}$ and $\nabla_{\dot{\gamma}} X = 0$. Define $Y(t) := \sin(\pi t/L) X$. Then



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\begin{aligned}
I(Y, Y) &= -\int_0^L g(Y, \nabla_{\dot{\gamma}}^2 Y + R(Y, \dot{\gamma})\dot{\gamma})dt \\
&= \int_0^L (\sin(\pi/L))^2(\pi^2/L^2 - g(R(X, \dot{\gamma})\dot{\gamma}, X))dt
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$\|X\|_g = 1$

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Let $L \geq \pi R$. Then $K \geq 1/R^2$ implies hat $I(Y,Y) \leq 0$.

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Assume γ has no conjugate points. Then the unique Jacobi field J with $J(0) = Y(0) = 0$ and $J(L) = Y(L) = 0$ (i.e. $J \equiv 0$) would satisfy

$$0 = I(J, J) < I(Y, Y) \leq 0$$

giving a contradiction.

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Hence γ has conjugate points and is not minimal. $\square$

Proof of Bonnet's Theorem

Symplectic Manifolds

Definition 92: Let M be a smooth manifold (without boundary, but not necessarily compact). A **symplectic structure** of **symplectic form** on M is a closed, non-degenerate 2-form $\omega \in \Omega^2(M)$, i.e. $d\omega = 0$ and for all $p \in M$

$$X \in T_pM \mapsto X \lrcorner \omega \in T_p^*M$$

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Lemma 93 : From the existence of a symplectic structure follows:

(1) The non-degeneracy implies that dim $M = 2n$ is even.

$V_0 = T_pM \ni v_1 \neq 0$ % $\omega_p(v_1, v_1) = 0$ $\qquad \omega_0 = \omega_p$

$(*) \Rightarrow \exists\, v_2 \notin \mathrm{span}(v_1) : \omega(v_1, v_2) = 1.$

w.r.t. $\mathcal{B}$: $\omega = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & 0 \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \;\; 1 \\ & & & & & -1 \;\; 0 \end{pmatrix}$

$\dim \mathrm{Ker}\, \omega(v_1, \cdot) = \dim \mathrm{Ker}\, \omega(v_2, \cdot) = \dim M - 1$

& $\mathrm{Ker}\, \omega(v_1, \cdot) \neq \mathrm{Ker}\, \omega(v_2, \cdot) \Rightarrow \boxed{\dim(\mathrm{Ker}\, \omega(v_1, \cdot) \cap \mathrm{Ker}(v_2, \cdot)) = \dim M - 2} = V_1$

$\omega_1 := \omega|V_1 \times V_1$ is non-deg. $\quad v \in V_1 : \omega(v, v_1) = \omega(v, v_2) = 0$

$(*) \Rightarrow \exists\, v' \in V_1 : \omega(v, v') \neq 0 \xRightarrow{\text{Induction}} (v_1, v_2, \ldots, v_{2n-1}, v_{2n})$ with

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(2) It is equivalent to

$$\omega^n = \omega \wedge \ldots \wedge \omega \neq 0$$

is a volume form. In particular, M has to be oriented.

$\mathcal{B}^* = (v_j^*) \subset T_p^*M$
is dual to $\mathcal{B}$

ω w.r.t. $\mathcal{B}$: $\omega = v_1^* \wedge v_2^* + v_3^* \wedge v_4^* + \ldots + v_{2n-1}^* \wedge v_{2n}^*$

$\omega^n = n!\, v_1^* \wedge v_2^* \wedge v_3^* \wedge \ldots \wedge v_{2n-1}^* \wedge v_{2n}^*$

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(3) If M is a closed manifold, then

$$b_2(M) := \dim H^2_{DR}(M) \geq 1.$$

Symplectic Manifolds

(3) $\omega^n \neq 0 \Rightarrow \int_M \omega^n > 0$

$d\omega = 0$ <u>Claim</u> $[\omega] \neq 0 \in H^2_{DR}(M)$

<u>Pf.</u>: Assume $\exists$ $\alpha \in \Omega^1(M)$ s.t. $d\alpha = \omega$, i.e. ω is exact.

Then $0 < \int_M \omega^n = \int_M (d\alpha)\wedge \omega^{n-1} = \int_M d(\alpha \wedge \omega^{n-1})$ $\quad d\omega = 0$

$$= \int_{\partial M = \emptyset} \alpha \wedge \omega^{n-1} = 0 \quad \lightning.$$

<u>Corollary</u>: $S^4, S^6, \ldots$ do not admit symplectic structures

Examples

(1) $\mathbb{R}^{2n} \cong T^*\mathbb{R}^n \cong \mathbb{C}^n$: The standard symplectic structure is given by

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With standard coordinates $(x_1, ..., x_n)$ on $\mathbb{R}^n$ and adapted coordinates $(p_1, ..., p_n, q_1, ..., q_n)$ where

$$(p_1, ..., q_n) \mapsto \theta_{(p_1,...,q_n)} := (q_1, ..., q_n, \sum_{k=1}^{n} p^k dq^k)$$

parametrizes $T^*\mathbb{R}^n$ so that with the projection $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$

$$\pi(\theta(p_1, ..., q_n)) = (q_1, ..., q_n).$$

and for the cordinate vector fields

$$\theta_{(p_1,...,q_n)}\Big(\frac{\partial}{\partial q_k}\Big) = p^k(p_1, ..., q_n) = p_k.$$

Examples

The identification is given by $(p_1, p_2, ..., p_n, q_1, q_2, ..., q_n) \mapsto (x_1, x_3, ..., x_{2n-1}, x_2, x_4, ..., x_{2n})$ and the symplectic form

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$$(x_1, x_2, \ldots, x_{2n-1}, x_{2n}) \mapsto (x_1 + \mathrm{i}x_2, \ldots, x_{2n-1} + \mathrm{i}x_{2n})$$

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Ex.

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Notice

$$\omega_{st} = \overset{\mathfrak{Im}}{\cancel{\Re e}}(\langle ., \rangle)$$

for the standard Hermitian product on $\mathbb{C}^n \cong T_z\mathbb{C}^n$, i.e. the **Kähler form**.

Examples

(2) Let $A \in \Omega^1(S^{2n+1}; \mathrm{i}\mathbb{R})$ be the connection one form on the total space of the Hopf bundle $S^{2n+1} \to \mathbb{C}P^n$ where the connection is given by the horizontal spaces $T_z^h S^{2n+1}$ which are the orthogonal complements to the orbits of the S^1-action

$$(z_1, \dots, z_{n+1}) \cdot g = (z_1 g, \dots, z_{n+1} g).$$

The curvature $F_A \in \Omega^2(\mathbb{C}P^n; \mathrm{i}\mathbb{R})$ defines the symplectic form

$$\omega_{FS} := -\mathrm{i} F_A,$$

called **Fubini Study form**.
(3) Let F be an oriented surface. Any **area form** on F defines a symplectic structure on F. $\Lambda^3_p F = 0 \quad \forall p$ since $\dim F = 2$.

Cotangent Bundles

Let Q be a smooth n-dimensional manifold. The **tautological one form** $\theta \in \Omega^1(T^*Q)$ is defined via

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Here $\pi : T^*M \to M$ is the (smooth) projection, $\alpha \in T_p^*M$ for some $p \in M$ and $X \in T_\alpha(T^*M)$. Then $d_\alpha\pi(X) \in T_pM$ and the expression on the right hand side makes sense.

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Proposition 94: θ is smooth and $d\theta \in \Omega^2(T^*M)$ is a symplectic form on T^*M. Exercise. Hint: express θ in adapted coordinate around α w.r.t. coordinates of Q around $\pi(\alpha) = p$.

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Proof:

Cotangent Bundles

Symplectomorphisms

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(2) Area-preserving diffeomorphism of oriented surfaces are symplectomorphisms. In particular, a linaer map $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a symplectomorphism if and only if $\det A = 1$. e.g. $A \in SO(2)$

or $A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$

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(3) Let $g : M \to N$ be a diffeomorphism ($\dim M = \dim N$). Then $\varphi : T^*N \to T^*M$ given by

$$\varphi: \quad \alpha \in T_p^*N \mapsto (d_{g^{-1}(p)}g)^*\alpha \in T_{g^{-1}(p)}^*M$$

is a symplectomorphism.

Hamiltonian Dynamics

Definition 96: A **Hamiltonian system** is a tripel (M, ω, H) where (M, ω) is a symplectic manifold, and $H : M \times \mathbb{R} \to \mathbb{R}$ is a smooth function. If $H : M \to \mathbb{R}$ i.e. independent on teh second $\mathbb{R}$-component, the system is called **autonomous**. H defines a $\mathbb{R}$-dependent vector field X_H on M via

$$\omega_p(X_H(p,t), Y) = -d_{p,t}H(Y)$$

for all $p \in M$ and $Y \in T_pM$, which is called **Hamiltonian vector field**, or with $H_t = H(., t) : M \to \mathbb{R}$ in short

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X_H uniquely defined by non-deg. of ω

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The **state** of the system is a point $x \in M$, its **dynamics** are the flow-lines $\gamma : I \to M$ of X_H:

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H is sometimes called Hamiltonian function – although it is still just an ordinary function.

Conservative Newtonian Mechanics

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Let $U : \Omega \times \mathbb{R} \subset \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ be the potential of a force field, i.e. the force acting on a point mass with mass m is given by

$$F = -\nabla U.$$

where derivatives are taken in space direction only. The point mass will move along curves $x : I \to \mathbb{R}^3$ which satisfy Newton's equations of motion:

$$\left(\frac{d}{dt}(m\dot{x}) = \right) \quad m\ddot{x}(t) = F(x(t), t) = -\nabla U(x(t), t)$$

$T^*\Omega$

Consider $(T^*\mathbb{R}^3, d\theta, H)$ with $H : T^*\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ given by

$$H(p, q) := \frac{1}{2m}\|p\|^2 + U(\cancel{q}x) \qquad \% \; p = m \cdot \dot{x}$$

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$$H(p, q) := \frac{1}{2m}\|p\|^2 + U(q).$$

The first summand is the **kinetic energy** the second the **potential energy**, H the **total energy** of the system.

Conservative Newtonian Mechanics

We compute $d^{T^*\mathbb{R}^3} H$

$$dH = \frac{1}{m} \sum_k p^k dp^k + \sum_k \frac{\partial U}{\partial q_k} dq^k$$

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With $\gamma(t) = (p(t), q(t))$ we find

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and with $q(t) = x(t), p(t) = m\dot{x}(t)$ we obtain Newton's equation again.

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(2) **Conservation of the symplectic structure:** Consider a general Hamiltonian system (M, ω, H). Let $\Phi : U \times (t_0 - \epsilon, t_0 + \epsilon) \to M$ be a smooth map for an open subset $U \subset M$ satisfying $\Phi(t_0, x) = x$ for all $x \in U$ and

$$\frac{\partial \Phi}{\partial t}(x, t) = X_H(\Phi(x, t), t)$$

also called the **flow of** X_H. We abbreviate $\Phi_t(X) := \Phi(x, t)$. Then $\Phi_t : U \to M$ is an embedding and

$$\Phi_t^* \omega = \omega.$$

Conservation Laws

(3) **Transformation under symplectomorphisms:** Let $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ be a symplectomorphism. Let $H : M_2 \times \mathbb{R} \to \mathbb{R}$ be a smooth function on M_2. Then for the Hamiltonian vector fields of H and $H \circ \varphi$

$$\varphi_*(X_{H\circ\varphi}) = X_H.$$

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Proof:

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Isotropic, Coisotropic and Lagrangian Immersions

For a symplectic vector space (V, ω) and a subspace $U \subset V$ we define

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Notice: If ι is isotropic, then $\dim N \leq \frac{1}{2}\dim M$, coisotropic, then $\dim N \geq \frac{1}{2}\dim M$. Hence, if ι is Lagrangian, then $\dim N = \frac{1}{2}\dim M$.

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Examples: (1) If $\dim N = 0$ or 1, then ι is isotropic. If $\dim N = n - 1$ or n, ι is coisotropic.

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(4) Let $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ be a symplectomorphism ($\dim M_1 = \dim M_2$). Then the graph

$$\Gamma_\varphi := \{(x, \varphi(x)) \mid x \in M_1\} \subset M_1 \times M_2$$

is a Lagrangian submanifold where the symplectic structure on $M_1 \times M_2$ is given by

$$\omega := \pi_1^*\omega_1 - \pi_2^*\omega_1.$$