Differential Geometry II

Darboux Theorem and Moser Trick

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Isotropic, Coisotropic and Lagrangian Immersions

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$$\mathrm{Ann}_\omega(U) := \{v \in V \mid \omega(v, u) = 0 \quad \forall u \in U\}.$$

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i.e. $v \in T_{\iota(p)}M \quad \omega(v,u) = 0 \quad \forall \; u \in \iota_*(T_pN)$

$\Rightarrow \; v \in \iota_*(T_pN)$.

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Lagrangian if isotropic and coisotropic.

Notice: If ι is isotropic, then $\dim N \leq \frac{1}{2} \dim M$, coisotropic, then $\dim N \geq \frac{1}{2} \dim M$. Hence, if ι is Lagrangian, then $\dim N = \frac{1}{2} \dim M$.

Isotropic, Coisotropic and Lagrangian Immersions

Examples: (1) If $\dim N = 0$ or 1, then ι is isotropic. If $\dim N = 2n-1$ or $2n$, ι is coisotropic.

$\omega(v,v) = 0 \ \forall v$

$\dim M = 2n$

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(3) The zero section and every fibre in T^*M are Lagrangian submanifolds. If $\alpha \in \Omega^1(Q)$ then its graph

$\rightarrow$ $\Gamma_\alpha := \{\alpha(q) \in T_q^*Q \mid q \in Q\} \subset T^*Q$ is Lagrangian if and only if $d\alpha = 0$.

$\omega = d\theta = \sum dp_i\, dq_i$ in associated coord. $(p_1, .., p_n, q_1, .., q_n)$

$\theta = \sum p_i\, dq_i$ $\quad \theta|_{\text{zero section}} \equiv 0$

$q_i|_{\text{fibre}} = \text{const.} \Rightarrow dq_i|_{\text{fibre}} \equiv 0 \Rightarrow \theta|_{T_q^*Q} \equiv 0$

$\alpha: Q \to T^*Q \quad \alpha^*\theta = \alpha \Rightarrow \alpha^*(d\theta) = d\alpha$

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(4) Let $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ be a symplectomorphism ($\dim M_1 = \dim M_2$). Then the graph

$$\Gamma_\varphi := \{(x, \varphi(x)) \mid x \in M_1\} \subset M_1 \times M_2$$

is a Lagrangian submanifold where the symplectic structure on $M_1 \times M_2$ is given by

$$\omega := \pi_1^*\omega_1 - \pi_2^*\omega_1. \quad (\text{"}= \omega_1 \oplus (-\omega_2)\text{"})$$

At (4) $\Phi: M_1 \to \Gamma_\varphi$ $\quad \Phi(x) = (x, \varphi(x))$.

$$\Phi^* \omega = (id \times \varphi)^* (\pi_1^* \omega_1 - \pi_2^* \omega_2)$$

$$= \omega_1 - \varphi^* \omega_2 = 0$$

.

Darboux' Theorem

Theorem 99: Let (M, ω) be a symplectic manifold, $\dim M = 2n$, $p \in M$. There exists a neighborhood U of p, $R > 0$ and a symplectomorphisms

$\mathbb{R}^{2n}$

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(2) The quantity

$$w(M, \omega) := \sup\{R > 0 \mid \exists \psi : B^{2n}(R) \hookrightarrow M, \psi^* \omega = \omega_{st}\}$$

is a symplectic invariant, called **Gromov width**. E.g.

$$w(B^2(R) \times \mathbb{R}^{2n-2}) = R.$$

Moser's Trick

Lemma 100: Let (M, ω) be a symplectic manifold.
(1) For a smooth family $(\omega_\tau)_{\tau \in [0,1]}$ of symplectic structures with $\omega_0 = \omega$ suppose there is a smooth family $\beta_\tau \in \Omega^1(M)$ such that

$$d\beta_\tau = \frac{d}{d\tau}\omega_\tau.$$

% $\nexists \beta_\tau$ if
% $\omega_\tau = e^\tau \omega$
% & M is closed.

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(2) Assume there is a family $\Phi_\tau : U \to M$ of diffeomorphisms onto their image such that $\Phi_0 = \mathrm{id}_U$, and

1st order ODE
with initial values

$$\frac{d}{d\tau}\Phi_\tau = X_\tau \circ \Phi_\tau$$

for the family of vector fields defined by

$X \mapsto \omega_p(X, .)$

$$\omega_\tau(X_\tau, .) = -\beta_\tau. \qquad \longleftarrow \quad T_pM \longrightarrow T_p^*M$$

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Then

$$\Phi_\tau^*\omega_\tau = \omega. \Rightarrow \Phi_1^*\omega_1 = \omega_0 = \omega$$

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(ii) Differentiating the left hand side of the equation yields

$$
\begin{aligned}
\frac{d}{d\tau}(\Phi_\tau^*\omega_\tau) &= \Phi_\tau^*(\mathcal{L}_{X_\tau}\omega_\tau + \frac{d\omega_\tau}{d\tau}) \\
&= \Phi_\tau^*(X_\tau \lrcorner d\omega_\tau + d(X_\tau \lrcorner \omega_\tau) + d\beta_\tau) \\
&= \Phi_\tau^*(d(-\beta_\tau) + d\beta_\tau) = 0.
\end{aligned}
$$

ω_τ symplectic $\Rightarrow d\omega_\tau = 0$

Cartan

The claim follows. □

Remark: The job consists in establishing the two conditions. To obtain β_τ one uses the idea of Poincaré'e Lemma or assumes that $[\omega_\tau] \in H^2_{DR}(M)$ is constant and tools from analysis.

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Read the proof of Darboux' Theorem in this light!!

Darboux Charts

historically different proof: see Arnol'd "Mathematical Methods..."

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(i) Let $(v_1, ..., v_{2n}) \subset \mathbb{R}^{2n}$ basis such that
$\omega_0(v_{2k-1}, v_{2k}) = -\omega_0(v_{2k}, v_{2k-1}) = 1$ and $\omega_0(v_i, v_j) = 0$ else.

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Let $T \in Gl(2k; \mathbb{R})$ such that $T(e_j) = v_j$. Then $T: T^{-1}(U) \to \cancel{T}(U)$ is a diffeomorphism with $(T^*\omega)_0 = \omega_{st}$.

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$\Rightarrow$ We may assume that $\omega_0 = \omega_{st}$.

$\omega_0 = \omega$, $\omega_1 = \omega_{st}$, $d\omega_\tau = 0$ $\forall \tau$

(ii) Let $\omega_\tau := (1-\tau)\omega + \tau\omega_{st}$. We have $\omega_{\tau,0} = \omega_{st}$ for all τ. $\Rightarrow$ There exists an open neighbourhood $U' \subset U$ of p such that

$$\omega_\tau\big|_{U'}$$

is non-degenerate for all $\tau \in [0,1]$.

Darboux Charts

$d\omega = 0$. U'' starshaped. Poincaré Lemma

(iii) PL $U'' \subset U'$ and $\beta \in \Omega^1(U'')$ such that $\beta_0 = 0$ and $d\beta = \omega$. Define

$$\beta_\tau := (1-\tau)\beta + \tau\theta$$

where $\theta := \sum_{k=1}^n x^{2k-1} dx^{2k}$ is the tautological form. $\beta_{\tau,0} = 0$ for all τ. Then $d\beta_\tau = \omega_\tau$.

$d\beta_\tau = (1-\tau)d\beta + \tau\, d\theta$
$= (1-\tau)\omega + \tau\,\omega_{st} = \omega_\tau$

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(iv) Let X_τ be the vector field on U'' such that

$$X\tau \lrcorner \omega_\tau = -\beta_\tau.$$

$X_\tau(0) = 0$ for all $\tau \in [0,1]$. Since $\beta_{\tau,0} = 0$.

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$X_\tau(0) = 0$ for all $\tau \in [0,1]$.

$\Rightarrow$ there is a neighbourhood $V \subset U''$ of p such that for all $q \in V$ there exists a unique solution $\gamma_q : [0,\cancel{t}\,1] \to U''$ of

$$\dot{\gamma}_q(t) = X_t(\gamma_q(t)) \quad \leftarrow$$

with $\gamma_q(0) = q$. Notice $\gamma_q \equiv p$ is a global solution

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(v) Define $\Phi_\tau : V \to U$ via $\Phi_\tau(q) = \gamma_q(\tau)$. Then

$$\Phi_1^* \overset{\omega_1}{\overset{\|}{\omega_{st}}} = \overset{\omega_0}{\overset{\|}{\omega}}$$

$$\frac{d}{d\tau}\Phi_\tau = X_\tau \circ \Phi_\tau. \quad \square$$

Almost Complex Structures

Definition 101: Let (M, ω) be a symplectic manifold.
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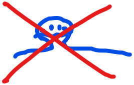

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(2) From $g(.,.) := \omega(., J.)$ follows that g is symmetric if and only if $g(JX, JY) = g(X, Y)$.
$\Rightarrow$ J is orthogonal, $h(X, Y) := g(X, Y) + \mathrm{i}\omega(X, Y)$ defines a Hermitian structure with Kähler form ω.

Almost Complex Structures

Proposition 102: Let M be a closed manifold, J an almost complex structure tamed by symplectic forms ω_k, $k = 0, 1$, such that $[\omega_0] = [\omega_1] \in H^2_{DR}(M)$. Then there is a symplectomorphism $\varphi : M \to M$, $\varphi^*\omega_1 = \omega_0$.

Proof: $\omega_1 - \omega_0 = d\beta$ $\quad \beta \in \Omega^1(M)$

$\omega_\tau = (1-\tau)\omega_0 + \tau\omega_1 = \omega_0 + \tau d\beta$

$\beta_\tau = \beta \quad : \quad d\beta_\tau = d\beta = \frac{d}{d\tau}\omega_\tau$

$X_\tau : \quad X_\tau \lrcorner\, \omega_\tau = -\beta$ smooth family of vector fields on M

M closed $\Rightarrow \exists\ \Phi_\tau : M \to M$ diffeo.

$\Phi_0 = \mathrm{id}_M, \quad \frac{d}{d\tau}\Phi_\tau = X_\tau \circ \Phi_\tau$

Moser's trick $\Rightarrow \Phi_1^*\omega_1 = \omega_0$ $\quad \square$

Almost Complex Structures

Theorem 103: (i) Let (M, ω) be a symplectic manifold. The space of compatible almost complex structures

$$\mathcal{J}(M, \omega) := \{J \mid J \text{ almost complex structure compatible with } \omega\}$$

is a non-empty contractible space.
(ii) Assume that on a open subset U there is an almost co~~p~~mplex structure tamed by ω such that $M \setminus U$ is compact, then there is an almost complex structure on M which is tamed by ω.

Proof: $J(T_pM, \omega_p) = \{ J \mid J$ complex str. of T_pM compatible with $\omega_p \}$

$Sp(T_pM, \omega_p) = \{ \phi : T_pM \to T_pM$ lin. iso $\mid \phi^*\omega_p = \omega_p \}$

$= Sp(T_pM, \omega_p) / U(T_pM, \omega_p, J_0)$

$\simeq \mathbb{R}^{N(n)}$ Siegel's upper half space.

$\leadsto$ form a bundle over M & $J(M, \omega) =$ space of sections $\Box$

Chern Classes of ω

Remark: The closedness $d\omega = 0$ plays no role in the proof of Theorem 103. The existence of an almost complex structure provides an obstruction to the existence of a symplectic structure.

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For example: An oriented closed 4–manifold admits an almost complex structure (inducing this orientation) if and only if there is a integer class $c \in H^2_{DR}(M)$ such that

$$\int_M c^2 = 2\chi(M) + 3\sigma(M).$$

S^4 and $2k\mathbb{C}P^2 \sharp 2\ell\overline{\mathbb{C}P^2}$ do not admit almost complex structures for any orientation.

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Definition 104: The **Chern classes** of a symplectic manifold (M, ω) are the Chern classes $c_k(TM, J)$ of an almost complex structure J which is compatible to ω.

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For example: An oriented closed 4–manifold admits an almost complex structure (inducing this orientation) if and only if there is a integer class $c \in H^2_{DR}(M)$ such that

$$\int_M c^2 = 2\chi(M) + 3\sigma(M).$$

S^4 and $2k\mathbb{C}P^2 \sharp 2\ell\overline{\mathbb{C}P^2}$ do not admit almost complex structures for any orientation.

Definition 104: The **Chern classes** of a symplectic manifold (M, ω) are the Chern classes $c_k(TM, J)$ of an almost complex structure J which is compatible to ω.

Remark: Since the space of such structures is connected via the Chern-Weil construction we see that $c_k(M, \omega)$ is well-defined, i.e. does not depend on J.

Proof of Theorem 103

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