Differential Geometry II

Holomorphic Curves

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Almost Complex Structures

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Theorem 103: (i) Let (M, ω) be a symplectic manifold. The space of compatible almost complex structures

$$\mathcal{J}(M, \omega) := \{J \mid J \text{ almost complex structure compatible with } \omega\}$$

is a non-empty contractible space.
(ii) Assume that on a open subset U there is an almost copmplex structure tamed by ω such that $M \setminus U$ is compact, then there is an almost complex structure on M which is tamed by ω.

Almost Complex Structures

Proof 2: The closedness $d\omega = 0$ plays no role in the proof.

1) (V,ω) ... symplectic vector space

g ... scalar product

$\exists!\ A \in Gl(V):\ g(v,w) = \omega(v, Aw)\ \forall\ v,w \in V$

claim: $A^* = -A$

pf: $g(Av,w) = \omega(Av, Aw) = -\omega(Aw, Av) = g(-Aw, v)$

$= g(v, (-A)w)$ □

$\Rightarrow (-A^2) \in Gl(V)$ symmetric & pos. definit

$\Rightarrow \exists\ \sqrt{-A^2} \in Gl(V)$ symmetric & pos. def.

$$\boxed{J = -A^{-1}\sqrt{-A^2}}$$

claim: (a) $\tilde{g}$ defined via $\tilde{g}(v,w) = \omega(v, Jw)$ is a scalar product

(b) $J^2 = -id_V$.

Almost Complex Structures

Pf: We show $\boxed{A^{-1}\sqrt{-A^2} = \sqrt{-A^2}\,A^{-1}}$: $-A^2$ is diagonalizable.

let $E_\lambda = \{ v \in V \mid -A^2 v = \lambda v \}$

$A^{-1}(-A^2) = -A = (-A^2)A^{-1} \Rightarrow A^{-1}(E_\lambda) = E_\lambda$

$v \in E_\lambda \quad (-A^2)(A^{-1}v) = A^{-1}(-A^2)v = \lambda(A^{-1}v)$

Construction of $\sqrt{-A^2}$: $E_\lambda = \{ v \in V \mid \sqrt{-A^2}\, v = \sqrt{\lambda}\, v \} \Rightarrow$ claim

a) $\omega(v, Jw) = g(v, A^{-1}w) = g(v, -A^{-2}\sqrt{-A^2}w) = g(-\sqrt{-A^2}A^{-2}v, w)$

$= g(-A^{-2}\sqrt{-A^2}v, w) = \omega(w, Jv) \Rightarrow \bar{g}$ is symmetric

$-A^2\sqrt{-A^2} = (\sqrt{-A^2})^3$ is symmetric with positive eigenvalues $\Rightarrow \bar{g}$ is positive definite

b) $J^2 = A^{-1}\sqrt{-A^2}\,A^{-1}\sqrt{-A^2} = A^{-2}(\sqrt{-A^2})^2 = A^{-2}(-A^2) = -Id$

2) M manifold, $\omega \in \Omega^2(M)$ pointwise non-degenerate

g Riemannian metric (through partition of unity) $\rightsquigarrow A \in \mathrm{Aut}(TM)$

$\overset{(1)}{\rightsquigarrow} J \in \mathrm{Aut}(TM)$ as required. $\square$

Chern Classes of ω

Remark: The existence of an almost complex structure provides an obstruction to the existence of a symplectic structure.

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For example: An oriented closed 4–manifold admits an almost complex structure (inducing this orientation) if and only if there is an integer class $c \in H^2_{DR}(M)$ such that $\int u^* c$ for $u : \Sigma \to M$ is even if and only if $u^* TM$ admits a spin structure (Σ a closed oriented surface) and

$$\int_M c^2 = 2\chi(M) + 3\sigma(M).$$

S^4 and $2k\mathbb{C}P^2 \sharp 2\ell \overline{\mathbb{C}P^2}$ do not admit almost complex structures for any orientation.

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Definition 104: The **Chern classes** of a symplectic manifold (M, ω) are the Chern classes $c_k(TM, J)$ of an almost complex structure J which is compatible to ω.

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Definition 104: The **Chern classes** of a symplectic manifold (M, ω) are the Chern classes $c_k(TM, J)$ of an almost complex structure J which is compatible to ω.

Remark: Since the space of such structures is connected, via the Chern-Weil construction we see that $c_k(M, \omega)$ is well-defined. i.e. does not depend on J.

Pseudoholomorphic Curves

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Definition 106: Let (M, J) be an almost complex manifold. A J–**holomorphic** (or **pseudoholomorphic**) curve is a Riemann surface (Σ, j) together with a map $u : \Sigma \to M$ such that for all $z \in \Sigma$

$$d_z u \circ j_p z = J_{u(z)} \circ d_z u.$$

Pseudoholomorphic Curves

In complex coordinates, $z = x + \mathrm{i}y$, this takes the form

$$\frac{\partial u}{\partial x}(z) + J(u(z))\frac{\partial u}{\partial y}(z) = 0.$$

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Proposition 107: Let h be a Hermitian metric on M, ω its Kähler form, $u : (\Sigma, j) \to (M, J)$ a J–holomorphic curve. Then $u^*\omega$ is compatible with the orientation of (Σ, j) wherever $d_z u \neq 0$.

Proof: $u^*\omega_z\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \omega_{u(z)}\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = \omega_{u(z)}\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) =$

$= \omega_{u(z)}\left(-J(u(z))\frac{\partial u}{\partial y}, \frac{\partial u}{\partial y}\right) = \left\|\frac{\partial u}{\partial y}\right\|^2_{g(u(z))} \geq 0$

& "=" $\Leftrightarrow$ $\frac{\partial u}{\partial y} = 0$ $\Leftrightarrow$ $d_z u = 0$ $\square$

Exact Lagrangian Submanifolds

Let (M, ω) be a symplectic manifold with an exact symplectic form, i.e. $\omega = d\alpha$ for a one form $\alpha \in \Omega^1(M)$ (e.g. $(\mathbb{C}^n, \omega_{st})$. If $L \subset M$ is a Lagrangian submanifold, then $\alpha|_{TL}$ defines a closed one form on L

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Definition 108: The Lagrangian L is called **exact** if $\alpha|_{TL}$ is exact. i.e. there exists a smooth function $f : L \to \mathbb{R}$ such that $df = \alpha|_{TL}$.

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Definition 108: The Lagrangian L is called **exact** if $\alpha|_{TL}$ is exact. i.e. there exists a smooth function $f : L \to \mathbb{R}$ such that $df = \alpha|_{TL}$.

Theorem 109 (Gromov): There exists no closed, exact Lagrangian submanifold in $(\mathbb{C}^n, \omega_{st})$.

Remark: This can be considered as a generalization of Jordan's Curve Theorem: $L \subset \mathbb{R}^2 = \mathbb{C}$ closed regular curve.

$\Rightarrow$ $\mathbb{C} \setminus L$ consists of two connected components, one is bounded; denote it by $\Omega \subset \mathbb{R}^2$. $\partial \Omega = L \Rightarrow \int_L \alpha = \int_\Omega d\alpha = \int_\Omega \omega = \text{area}(\Omega) > 0$

if $\alpha = df$ for $f : L \to \mathbb{R}$ then $\int_L \alpha = \int_L df = \int_{\partial L = \emptyset} f = 0$ ↯

Exact Lagrangian Submanifolds

Proof: Gromov: $\exists\ u: \Delta \subset \mathbb{C} \to \mathbb{C}^2$ $\quad \Delta = \{z \in \mathbb{C} \,/\, |z| \le 1\}$

holomorphic, non-constant s.t.

$u(\partial\Delta) \subset L$.

$$\overset{\text{Prop 10.3}}{\Rightarrow}\ 0 < \int_\Delta u^*\omega = \int_\Delta u^*(d\alpha) = \int_\Delta d(u^*\alpha)$$

$$= \int_{\partial\Delta} u^*\alpha$$

If $\alpha|_{TL} = df$ for $f: L \to \mathbb{R}$ smooth

$$\int_{\partial\Delta} u^*\alpha = \int_{\partial\Delta} u^*(df) = \int_{\partial\Delta} d(f \circ u) = \int_{\partial(\partial\Delta) = \emptyset} f \circ u = 0 \ \text{⚡} \quad \square$$

Non-Squeezing

Theorem 110 (Gromov): Let $Z^{2n}(R) := B^2(R) \times \mathbb{R}^{2n-2} \subset \mathbb{R}^{2n}$ and $B^{2n}(r) \subset \mathbb{R}^{2n}$ be equipped with the standard symplectic structure. Assume there is an embedding

$$\varphi : B^{2n}(r) \subset Z^{2n}(R)$$

which is a symplectomorphism onto its image: $\varphi^* \omega_{st} = \omega_{st}$. Then $r \leq R$.

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Corollary: The group of symplectomorphisms, $\text{Symp}_c(\mathbb{R}^{2n})$, of $\mathbb{R}^{2n}$ with compact support is not dense in C^0-topology in the group of volume preserving diffeomorphisms with compact support.

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Corollary: The group of symplectomorphisms, $\mathrm{Symp}_c(\mathbb{R}^{2n})$, of $\mathbb{R}^{2n}$ with compact support is not dense in C^0-topology in the group of volume preserving diffeomorphisms with compact support. Let $\varphi_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a volume preserving diffeomorphism such that $\varphi_0(B^{2n}(2R)) \subset Z^{2n}(R)$. Then there is no sequence $(\varphi_k)_k \subset \mathrm{Symp}_c(\mathbb{R}^{2n})$ such that $\varphi_k \to \varphi_0$ uniformly (in C^0).

∃ linear volume preserving maps

Non-Squeezing

$\forall \varepsilon > 0$

If there was such sequence $\exists\, k_0 : \forall k \geq k_0 \ \|(\varphi_k - \varphi_0)|_{B^{2n}(\frac{3R}{2})}\|_{C^0} < \varepsilon$.

$\varepsilon < \operatorname{dist}(\varphi_0(B^{2n}(\frac{3R}{2})), \partial Z^{2n}(R)) \Rightarrow \varphi_k(B^{2n}(\frac{3R}{2})) \subset Z^{2n}(R)$ ↯

Remark: Last statement indicates existence of symplectic topology:

Gromov's h-principle: M^{2n} closed, connected, $\eta \in \Omega^2(M)$ non deg, $a \in H^2_{dR}(M)$.

$n \geq 2$ s.t. $a^n \in H^{2n}_{dR}(M)$ volume form

$\exists$ symplectic structure $\omega \in \Omega^2(M \setminus \{p\})$ $p \in M$ arbitrary

- $\omega \simeq \eta|_{M \setminus \{p\}}$
- $[\omega] = a|_{M \setminus \{p\}}$

academic discussion: If corollary was not true, one could extend ω to all of M! Non-trivial fact. Most examples found later!

However: There are M which admit η & a as above for which there exists no symplectic structure!

Monotonicity

For the proof of Theorem 110 we will need the following

Proposition 111: Let $u : (\Sigma, j) \to \mathbb{C}^n$ be a holomorphic curve (in Algebra: "complex curve"), $u(p) = 0$ for $p \in \Sigma$ and $r > 0$ such that $u^{-1}(B^{2n}(r)) \subset \Sigma$ is compact. Then

Monotonicity

For the proof of Theorem 110 we will need the following

Proposition 111: Let $u : (\Sigma, j) \to \mathbb{C}^n$ be a *non-constant!* holomorphic curve (in Algebra: "complex curve"), $u(p) = 0$ for $p \in \Sigma$ and $r > 0$ such that $u^{-1}(B^{2n}(r)) \subset \Sigma$ is compact. Then

$$\text{area}(u(\Sigma)) \geq \pi r^2.$$

Sketch of proof:
- w.l.o.g. Σ is somewhere injective.
- pick $r' > r$ s.t. $d_z u \neq 0 \; \forall z \in u^{-1}(\partial B^{2n}(r'))$

s.t. $\Sigma_s := u^{-1}(\overline{B^{2n}(s)})$, Σ compact, $\partial \Sigma_s = \partial B^{2n}(s)$

w.l.o.g. Σ connected.

$\Rightarrow \text{area}(u(\Sigma_{r'})) = \int_{\Sigma_{r'}} u^* \omega_{st} = \int_{\partial \Sigma_{r'}} u^* \alpha$

for $\alpha = \frac{1}{2} \sum (x_k \, dy_k - y_k \, dx_k)$ ($\Rightarrow d\alpha = \omega_{st}$)

$\alpha_0 \in \Omega^1(\mathbb{C}^n \setminus 0)$ $\quad \alpha_0 = \pi^* \alpha$ $\quad \pi : \mathbb{C}^n \setminus 0 \to \partial B^{2n}(r')$ $\quad \pi(z) = r' \frac{z}{|z|}$.

• we have $d\alpha_0(\cdot, J\cdot)$ pos. semi definite.

$$\Rightarrow 0 \leq \int_{\Sigma_{r'} \setminus \Sigma_s} u^* d\alpha_0 = \int_{\partial \Sigma_{r'}} u^* \alpha_0 - \int_{\partial \Sigma_s} u^* \alpha_0$$

• $\exists\, x \in u(\Sigma)$ close to 0 s.t. : • $u^{-1}(x)$ consists of 1 point p'

• $d_{p'} u \neq 0$.

replace p by p' & u by $u - u(p')$.

__Then:__
$$\lim_{s \to 0} \int_{\partial \Sigma_s} u^* \alpha_0 = \int_{\partial \Delta} u_0^* \alpha_0 = (r')^2 \int_0^{2\pi} \frac{d\theta}{2} = \pi (r')^2 \geq \pi r^2$$

$u_0 : \Delta \to \mathbb{C}^n$, $u_0(z) = (\lambda_1 z_1, .., \lambda_n z_n)$, $\lambda := d_p u\left(\begin{pmatrix}1\\0\end{pmatrix}\right)$.

$$\Rightarrow \int_{\Sigma} u^* \omega \geq \int_{\partial \Sigma_{r'}} u^* \alpha \geq \pi r^2$$

$\square$

Proof of Non-Squeezing

Embedd $\iota : (Z^{2n}(R), \omega_{st}) \hookrightarrow (S^2(R) \times \mathbb{R}^{2n-2}, \omega) =: (M, \omega)$ with $\omega = \pi_1^*\omega_R + \pi_2^*\omega_{st}$, where $\omega_R \in S^2$ area form with

$$\int_S^2 \omega_R = \pi R^2.$$

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Sufficient to show that $\varphi : (B^{2n}(r), \omega_{st}) \hookrightarrow (M, \omega)$ implies $r \leq R$.

Remark: If $r - \varepsilon \leq R$ for any $\varepsilon > 0$ claim follows.

Method of construction of ω-compatible J, allows to construct such J on M s.t.

$$J|_{\varphi(B^{2n}(r-\varepsilon))} = \varphi_* J_{st}$$

$$J|_{M \setminus \varphi(B^{2n}(r))} = J_{st}$$

Gromov: $\exists$ J-hol. $u : S^2 \to M$ s.t.

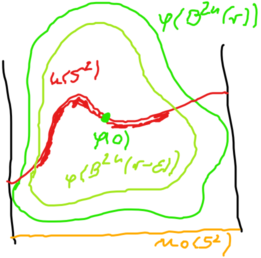
$$u(N) = \varphi(0).$$

$$u \simeq u_0 : S^2 \to M$$

$$u_0(z) = (z, v)$$

$v \in \mathbb{R}^{2n-2}$ s.t. $u_0(S^2) \subseteq M \setminus \varphi(B^{2n}(r))$

$\Rightarrow$ u_0 J-hol. & $\int_{S^2} u^*\omega = \int_{S^2} u_0^*\omega = \pi R^2.$



Apply Prop 111 to

$\varphi^{-1} \circ u \big|_{u^{-1}(\varphi(B^{2n}(r-\varepsilon)))}$ $\qquad \varphi^*\omega = \omega$

$$\Rightarrow \int_{S^2} u^*\omega \geq \int_{u^{-1}(\varphi(B^{2n}(r-\varepsilon)))} u^*\omega \overset{\downarrow}{=} \int_{u^{-1}(\varphi(B^{2n}(r-\varepsilon)))} (\varphi^{-1} \circ u)^*\omega$$

$$\geq \pi (r-\varepsilon)^2 \Rightarrow R \geq r - \varepsilon \qquad \square$$