Differential Geometry II

Differential Forms

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Differential Forms on $\mathbb{R}^n$

Definition 8: Let $U \subset \mathbb{R}^n$ be an open subset. A **(differential)** k**–form** on U is a differentiable map $\alpha : U \longrightarrow \Lambda^k((\mathbb{R}^n)^*)$, i.e. the components α_I

$$\alpha_p = \sum_{I=\{1\leq i_1<i_2<\ldots<i_k\}} \alpha_I(p)\underline{e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_k}}$$

where $\{e^1, \ldots, e^n\}$ denotes the basis dual to the standard basis are smooth functions (or C^m–functions etc.). We typically write α_p instead of $\alpha(p)$. The $\alpha_I : U \to \mathbb{R}$ are differentiable.

The space of such forms will be denoted by $\underline{\Omega^k(U)}$.

Differential Forms on $\mathbb{R}^n$

Examples: (1) Let $X : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field. Then $\langle X(.), . \rangle$ defines a differential one form.
(2) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Then its differential df defines also a one form. We have

$$d_p f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e^i.$$

$x^i(x_1, \dots, x_n) = x_i$

In particular, for the i–th coordinate function x^i we see that $dx^i = e^i$. Therefore and since the the e^i are closely related to the choice of coordinates, we will from now on write dx^i instead of e^i:

$$d_p f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx^i.$$

Wedge-Product and Inner product

(1) Given a differential k–form $\alpha \in \Omega^k(U)$ and a differential ℓ–form $\beta \in \Omega^\ell(U)$, the wedge-product $\alpha \wedge \beta \in \Omega^{k+\ell}(U)$ is defined by pointwise applying the wedge-product:

$$(\alpha \wedge \beta)_p := \underline{\alpha_p \wedge \beta_p}. \quad \text{last time}$$

Check that the Result is a differential form, i.e. smooth.

(2) Given a smooth vector field X on U and a differential form $\alpha \in \Omega^k(U)$, then the inner product is also defined pointwise:

$$(X \lrcorner \alpha)_p := \underline{X_p \lrcorner \alpha_p} \quad \text{last time}$$

giving rise to a differential $(k-1)$–form (Check that!)

Pull-Back

(3) For a smooth map $F : U \to V$, where $V \subset \mathbb{R}^m$ open and a differential k–form $\alpha \in \Omega^k(V)$, its pull-back, $F^*\alpha$ is defined via

$$p \in U \qquad (F^*\alpha)_p := (d_pF)^*\alpha_{F(p)} \in \Lambda^k((\mathbb{R}^m)^*)$$

Once again, $F^*\alpha \in \Omega^k(U)$, i.e. smooth (Check it!).

Examples: (1) $g \in \Omega^0(V) = C^\infty(V)$: $F^*g = g \circ F$.
(2)$F^*(dx^j) = \sum_{i=1}^{n} \cancel{n} \frac{\partial F^j}{\partial x_i} dx^i = dF^j$ / $F = (F^1, F^2, \dots, F^m)$
(3) $m = n : F^*(dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n) = \underbrace{\det(dF)}_{\text{Jacobian}} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$.

Proposition 9: The pull-back is a homomorphism of graded algebras over $\mathbb{R}$.

$$\Omega^*(U) = \bigoplus_{k=0}^{n} \Omega^k(U)$$

$f \in C^\infty(V)$, $\alpha \in \Omega^k(V)$
$f \cdot \alpha \in \Omega^k(V)$
$F^*(f \cdot \alpha) = F^*(f) \cdot F^*(\alpha)$

Proof: Exercise.

Notice that $\Omega^*(U), \Omega^*(V)$ are algebras over $C^\infty(U)$, $C^\infty(V)$, respectively, but even if $U = V$ the pull-back is not a algebra homomorphism w.r.t. that structure.

The Exterior Derivative

Intuition: 1) $f \in \Omega^0(U) = C^\infty(U)$
then $df \in \Omega^1(U)$ is the differential

2) $\alpha \in \Omega^k$, $\beta \in \Omega^l$
$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$

Definition 10: For

$$\alpha = \sum_{I=\{1\leq i_1<i_2<\ldots<i_k\}} \alpha_I dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} \in \Omega^k(U)$$

$\leq n$

its **exterior derivative**, $d\alpha \in \Omega^{k+1}(U)$ is defined via

$$d\alpha = \sum_{I=\{1\leq i_1<i_2<\ldots<i_k\}} \sum_{j=1}^{n} \frac{\partial \alpha_I}{\partial x_j} dx^j \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}.$$

Notice that the summand vanishes for those I and j for which $j \in I$.

$\mathbb{R}^n$

Examples: (1) If $\alpha \in \Omega^n(U)$, then $d\alpha = 0$ by the remark at the end.
(2) For $\alpha = x_1 dx_2 \in \Omega^1(\mathbb{R}^2)$: $d\alpha = dx_1 \wedge dx_2$.

The Exterior Derivative

(3) Let $f \in C^\infty(U)$. Then

$$d(df) = d\Big(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx^i\Big) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} dx^j \wedge dx^i.$$

Now, $dx^i \wedge dx^j = -dx^j \wedge dx^i$ hence all summands with $i = j$ vanish and the sum reduces to

$$= \sum_{1 \le i < j \le n} \Big(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\Big) dx^i \wedge dx^j = 0$$

due to Schwarz' Lemma. $\square$

The Exterior Derivative

Theorem 11: $(\Omega^*(U), \wedge, d)$ is a **differential graded algebra** over $\mathbb{R}$, meaning that $(\Omega^*(U), \wedge)$ is an algebra over $\mathbb{R}$, d increases the degree by 1, is $\mathbb{R}$-linear and (1) d satisfies Leibniz' rule:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

for $\alpha \in \Omega^k(U), \beta \in \Omega^\ell(U)$
(2) $$d^2 = d \circ d = 0.$$
(3) Moreover, if $F : U \to V$ is a smooth map as before the pull-back defines a morphism~~s~~ of differential graded algebras, i.e. in particular for $\alpha \in \Omega^k(V)$ we have

$$d(F^*\alpha) = F^*(d\alpha).$$

Proof of Theorem 11 (1):

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

$$\Omega^k \ni \alpha = \sum_{I=\{1 \le i_1 < \dots < i_k \le n\}} \alpha_I \, dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$\beta = \sum_{J=\{1 \le j_1 < \dots < j_l \le n\}} \beta_J \, dx^{j_1} \wedge \dots \wedge dx^{j_l}$$

$$d(\alpha \wedge \beta) = d\left(\sum_I \sum_J \alpha_I \beta_J \, dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}\right)$$

$$= \sum_I \sum_J \sum_{m=1}^n \left(\frac{\partial \alpha_I}{\partial x_m} \beta_J + \alpha_I \frac{\partial \beta_J}{\partial x_m}\right) dx^m \wedge dx^{i_1} \wedge \dots \wedge dx^{j_l}$$

$$= \sum_I \sum_J \sum_{m=1}^n \frac{\partial \alpha_I}{\partial x_m} \beta_J \, dx^m \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \, dx^{j_1} \wedge \dots \wedge dx^{j_l}$$

$$+ \sum_I \sum_J \sum_{m=1}^n (-1)^k \alpha_I \frac{\partial \beta_J}{\partial x_m} \, dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^m \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}$$

$$= d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

since

$$d\alpha = \sum_I \sum_{m=1}^{n} \frac{\partial \alpha_I}{\partial x_m} dx^m \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$d\beta = \sum_J \sum_{m=1}^{n} \frac{\partial \beta_J}{\partial x_m} dx^m \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}$$

Proof of Theorem 11 (2): $d \circ d = 0$

$$\alpha = \sum_{I = \{1 \le i_1 < \dots < i_k \le n\}} \alpha_I \, dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$d\alpha = \sum_{I} \sum_{j=1}^{n} \frac{\partial \alpha_I}{\partial x_j} dx^{j} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$d(d\alpha) = \sum_{I} \sum_{j=1}^{n} \sum_{m=1}^{n} \frac{\partial^2 \alpha_I}{\partial x_m \partial x_j} dx^{m} \wedge dx^{j} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$= \sum_{I} \sum_{j<m} \underbrace{\left(\frac{\partial^2 \alpha_I}{\partial x_m \partial x_j} - \frac{\partial^2 \alpha_I}{\partial x_j \partial x_m} \right)}_{= 0 \text{ Schwarz' Lemma}} dx^{m} \wedge dx^{j} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$= 0$$

Proof of Theorem 11 (3): $F^* \circ d = d \circ F^*$

$$\alpha = \sum_{I = \{1 \le i_1 < i_2 < \dots < i_k \le m} \alpha_I \, dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Omega^k(V)$$

$\mathbb{R}^m$

$$d\alpha = \sum_I \sum_{j=1}^m \frac{\partial \alpha_I}{\partial x_j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$F : \mathcal{U} \subset \mathbb{R}^n \to V$ smooth

$$F^*(d\alpha) = \sum_I \sum_{j=1}^m \left(\frac{\partial \alpha_I}{\partial x_j} \circ F\right) F^* dx^j \wedge F^* dx^{i_1} \wedge \dots \wedge F^* dx^{i_k}$$

$$= \sum_I \sum_{j=1}^m \sum_{J = \{j_1, j_2, \dots j_{k+1}\}} \left(\frac{\partial \alpha_I}{\partial x_j} \circ F\right) \left(\frac{\partial F^j}{\partial x_{j_1}} dx^{j_1}\right) \wedge \left(\frac{\partial F^{i_1}}{\partial x_{j_2}} dx^{j_2}\right) \wedge \dots \wedge \left(\frac{\partial F^{i_k}}{\partial x_{j_{k+1}}} dx^{j_{k+1}}\right)$$

$$d(F^*\alpha) = d\left(\sum_I \sum_{J=\{j_2,j_3,\dots,j_k\}} (\alpha_I \circ F) \left(\frac{\partial F^{i_1}}{\partial x_{j_2}} dx^{j_2} \right) \wedge \dots \wedge \left(\frac{\partial F^{i_k}}{\partial x_{j_{k+1}}} dx^{j_{k+1}} \right) \right)$$

chain rule

$$= \sum_I \sum_{J=\{j_1,j_2,\dots,j_{k+1}\}} \sum_{j=1}^{m} \left(\frac{\partial \alpha_I}{\partial x_j} \circ F \right) \left(\frac{\partial F^j}{\partial x_{j_1}} dx^{j_1} \right) \left(\frac{\partial F^{i_1}}{\partial x_{j_2}} dx^{j_2} \right) \wedge \dots \wedge \left(\frac{\partial F^{i_k}}{\partial x_{j_{k+1}}} dx^{j_{k+1}} \right)$$

$$+ \sum_I \sum_J (\alpha_I \circ F) \left\{ \left(\frac{\partial^2 F^{i_1}}{\partial x_{j_1} \partial x_{j_2}} dx^{j_1} \wedge dx^{j_2} \right) \wedge \dots + \frac{\partial F^{i_1}}{\partial x_{j_2}} dx^{j_2} \wedge \left(\frac{\partial^2 F^{i_2}}{\partial x_{j_1} \partial x_{j_3}} dx^{j_1} \wedge dx^{j_3} \right) \wedge \dots \right\} = 0$$

Schwarz' Lemma.

$\square$

Poincaré's Lemma

Definition 12: A k–form $\alpha \in \Omega^k(U)$ is called **closed** or **cocycle**, if $d\alpha = 0$, it is called **exact** or **coboundary** if there exists $\beta \in \Omega^{k-1}(U)$ such that $d\beta = \alpha$. The set of coboundaries, $B^k(U)$, and of cocycles, $Z^k(U)$, are linear subspaces of the set of k–Forms, i.e. $B^k(U) \subset Z^k(U) \subset \Omega^k(U)$. The quotient

$$H^k_{DR}(U) := Z^k(U)/B^k(U).$$

is called **de Rham cohomology** of U.

$U = \mathbb{R}^2 \setminus \{0\}$

$H^1_{DR}(U) \ni [\alpha] \mapsto \int \alpha \in \mathbb{R}$

$\alpha \in \Omega^1(U),\ d\alpha = 0$

S^1

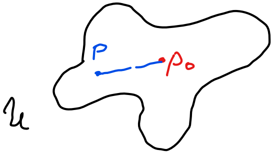
Examples: (1) $H^0_{DR}(U) \cong \mathbb{R}$ if U is connected and freely generated by its connected components in general.
(2) $U \subset \mathbb{R}^n$: $H^n_{DR}(U) = \{0\}$.
Exercise.

note: $Z^n(U) = \Omega^n(U) = B^n(U)$

clear

harder: easy for $U = \mathbb{R}^n$

Poincaré's Lemma



Definition 13: A subset $U \subset \mathbb{R}^n$ is called **starshaped** if there exists a point $p_0 \in U$ such that for all $p \in U$ all points of the segment between p_0 and p are contained in U.

Theorem 14: Let $U \subset \mathbb{R}^n$ be an open, starshaped set. Then for every $k > 0$, every closed k–Form $\alpha \in \Omega^k(U)$ is exact, i.e. there exists a $(k-1)$–form $\beta \in \Omega^{k-1}(U)$ such that $d\beta = \alpha$. In particular,

$$H^k_{DR}(U) = \{0\}$$

for every $k > 0$.

Chain Homotopies

Poincaré's Lemma is a special case of a much more general statement. Let $f, g : U \to V$ be two differentiable maps between open subsets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, $\Phi : U \times [0,1] \to V$ differentiable such that $\Phi(x,0) = f(x)$ and $\Phi(x,1) = g(x)$ for all $x \in \cancel{V}\,\mathcal{U}$.

Example: $U \subset \mathbb{R}^n$ open and starshaped w.r.t. p_0. $f := id_U : \mathcal{U} \to \mathcal{U}$
$g := c_{p_0} \equiv p_0$ and $\Phi(x,t) = p_0 + (1-t)(x - p_0)$, $\Phi : \mathcal{U} \times [0,1] \to \mathcal{U}$

Theorem 15: There exists linear maps $P_k : \Omega^k(V) \to \Omega^{k-1}(U)$ such that for all k

$$P_{k+1} \circ d + d \circ P_k = f^* - g^* : \Omega^k(V) \to \Omega^k(U).$$

Proof of Poincaré's Lemma

Theorem 15 applied to example and noting $c_{p_0}^*\alpha = 0$ for any k–form α with $k > 0$ yields:

$\alpha \in \Omega^k(U)$ closed, i.e. $d\alpha = 0$. Then

$$id^*\alpha - c_{p_0}^*\alpha = \alpha = \underbrace{P_{k+1}(d\alpha)}_{=0} + d(P_k(\alpha)) = d(P_k(\alpha))$$

so, $\beta := P_k(\alpha) \in \Omega^{k-1}(U)$ gives the desired $(k-1)$–form.

Proof of Theorem 15

$$\Phi : \mathcal{U} \times [0,1] \to V, \quad \alpha \in \Omega^k(V) \mapsto \Phi^*\alpha \in \Omega^k(\mathcal{U}\times[0,1])$$

$$P_k(\alpha) := \int_0^1 \underbrace{i_t^*\left(\frac{\partial}{\partial t} \lrcorner \Phi^*\alpha\right)}_{=:P_t(\alpha) \in \Omega^{k-1}(\mathcal{U})} dt$$

$i_t : \mathcal{U} \to \mathcal{U}\times[0,1]$

$x \mapsto (x,t)$

$\phi_t := \Phi \circ i_t : \mathcal{U} \to V$

i.e. $\phi_t(x) = \Phi(t,x)$

$$\Rightarrow \quad (\Phi^*\alpha)_{x,t} = (\phi_t^*\alpha)_x + dt \wedge P_t(\alpha)_x$$

$$\phi^*\alpha_{(x,t)}\left(\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_k}}\right) = \alpha_{x,t}\left((\phi_t)_* \frac{\partial}{\partial x_{i_1}}, \dots, (\phi_t)_* \frac{\partial}{\partial x_{i_k}}\right)$$

$$\phi^*\alpha_{(x,t)}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_{j_1}}, \dots, \frac{\partial}{\partial x_{j_{k-1}}}\right) = \alpha_{x,t}\Big(\underbrace{\phi_*\left(\tfrac{\partial}{\partial t}\right)}_{= \frac{\partial \phi}{\partial t}(x,t)}, \phi_{t*}\frac{\partial}{\partial x_{j_1}}, \dots, \Big)$$

Proof of Theorem 15

$$P_k(\alpha) = \int_0^1 \rho_t(\alpha)\, dt \quad \in \Omega^{k-1}(k)$$

$$d\, P_k(\alpha) = d \int_0^1 \rho_t(\alpha)\, dt = \int_0^1 d\, \rho_t(\alpha)\, dt$$

$$\phi^*(d\alpha) = d(\Phi^*\alpha) = d(\Phi_t^*\alpha) - dt \wedge d\, \rho_t(\alpha) \quad \Longleftarrow$$

$$P_{k+1}(d\alpha) = \int_0^1 i_t^*\left(\frac{\partial}{\partial t} \lrcorner\, \Phi^*(d\alpha)\right) dt = -\int_0^1 d\, \rho_t(\alpha)\, dt + \int_0^1 \frac{\partial (i_t^* \Phi_t^* \alpha)}{\partial t}\, dt$$

$$= -\int_0^1 d\, \rho_t(\alpha) + i_1^* \Phi_1^* \alpha - i_0^* \Phi_0^* \alpha$$

$$= -\int_0^1 d\, \rho_t(\alpha) + f^*\alpha - g^*\alpha$$

Proof of Theorem 15

$$d(P_k(\alpha)) = + \underbrace{\int_0^1 d\,P_t(\alpha)}$$

$\square$