Differential Geometry II

Applications of Stokes'Theorem

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Brouwer's Fixed Point Theorem

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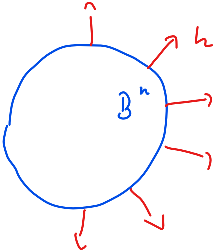
the **outer normal** given by $\mathbf{n}(x) = x$.



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defines a positive ~~mass~~ on S^{n-1}. Hence $d\alpha = 0$ and
measure

$$\int_{S^{n-1}} \alpha > 0.$$

Proof of Brouwer's Fixed Point Theorem

But by Stokes' Theorem and $f|_{S^{n-1}} = id_{S^{n-1}}$

$$0 = \underbrace{\int_{B^n} d(f^*\alpha)} = \int_{S^{n-1}} f^*\alpha = \int_{S^{n-1}} \alpha > 0$$

and we arrive at a contradiction.

$d(f^*\alpha) = f^*(\underbrace{d\alpha}_{=0}) = 0$

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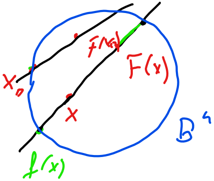
Proof of Theorem 31: (1) Assume there is a smooth function $F : B^n \to B^n$ without fixed points.
Define $f : B^n \to S^{n-1}$ which assigns to $x \in B^n$ the intersection of the well-defined line through x and $F(x)$, such that $x \in [f(x), F(x)]$. Then

$$f|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}.$$



Exercise: f is smooth.
This contradicts Proposition 32.

Proof of Brouwer's Fixed Point Theorem

(2) Assume there is a continuous function $F : B^n \to B^n$.
Approximate F by smooth functions $F_n : B^n \to B^n$ which converge uniformly to F (How?).

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Approximate F by smooth functions $F_n : B^n \to B^n$ which converge uniformly to F (How?).

Let $x_n \in B$ be fixed points: $F_n(x_n) = x_n$.
B compact: subsequence converges in B^n: $x_n \to x_* \in B^n$. Then

$$F(x_*) = x_*. \quad \square$$

(Exercise).

Riemannian Metrics

Recall: A **Riemannian metric** on a smooth manifold with boundary is a smooth family $\{g_p\}_{p \in M}$ of symmetric positive definite bilinear forms on T_pM.

$\left\{\frac{\partial}{\partial x_k}\right\} \rightsquigarrow (g_{ij}) : V \to M(n, \mathbb{R})$

Proposition 33: Any separable smooth manifold with boundary admits a Riemannian metric.

For this we will discuss the partition of unity.

Partition of Unity

Lemma 34: Let M be a smooth manifold with boundary. Let $\{U_\iota\}_{\iota \in I}$ be an open covering of M. There exist a countable family $\{\lambda_k\}_{k \in \mathbb{N}}$ of non-negative smooth functions with compact support, such that

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(iii) **partition of unity:**

$$\sum_{k=1}^{\infty} \lambda_k \equiv 1.$$

We will discuss the proof later, possibly.

Construction of Riemannian Metric

Proof of Proposition 33: Let $\{(U_\iota, \varphi_\iota, V_\iota)\}_{\iota \in I}$ be a smooth atlas of M, $\dim M = n$ and let $\{\lambda_k\}_{k \in \mathbb{N}}$ be a partition of unity w.r.t. covering. Let $\iota_k \in I$, such that $\operatorname{supp}(\lambda_k) \subset U_{\iota_k}$.

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Define for any $p \in M$

$$g_p := \sum_{k=1}^{\infty} \lambda_k(p)(\varphi_{\iota_k}^{-1})_p^* \langle ., . \rangle$$

where $\langle ., . \rangle$ denotes the standard scalar product in $\mathbb{R}^n$.

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$(\varphi_{\iota_k}^{-1})^* \langle ., . \rangle$ is symmetric positive definite.
g_p is is finite convex linear combination of such, hence symmetric and positive definite. $\square$

The Volume Form

Let M be an *oriented* n–dimensional manifold equipped with a Riemannian metric g.

Definition 35: The **volume form** of (M, g) is the n–form, dM, which is given at any $p \in M$ by the volume form of the oriented euclidean vector space

$$dM_p := d(T_pM, g_p).$$

See lecture 1.

dM is not (in general) the differential of a $(n-1)$-form on M : $dM \neq d(M)$

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Let (U, φ, V) be an oriented chart. Then

$$\varphi^*(dM) = \sqrt{\underbrace{\det(g)}_{>0}}\, dx^1 \wedge \ldots \wedge dx^n.$$

(Exercise).

The Volume Form

In particular dM defines a positive measure. We define the **volume** of (M, g) as

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Corollary: Assume $\partial M = \emptyset$ and compact, and oriented. Then the de Rham chomology class of dM is non-zero

$$[dM] \neq 0 \in H^n_{DR}(M).$$

Proof: Assume $dM = d\alpha$ for some $\alpha \in \Omega^{n-1}(M)$

$$0 < \int_M dM = \int_M d\alpha \underset{\text{Stokes}}{=} \int_{\underset{\emptyset}{\underset{\shortparallel}{\partial M}}} \alpha = 0 \quad \lightning.$$

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Corollary: Assume $\partial M = \emptyset$. Then the de Rham chomology class of dM is non-zero

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If $\mathbf{n}$ is the outward normal vector field of M along ∂M, then

$n_p \perp T_p(\partial M)$

$\|n_p\|_{g_p} = 1$

$$d(\partial M) = n \lrcorner dM.$$

(Exercise).

Obstruction for Retract to the Boundary

Theorem 36: Let M be a compact, oriented manifold with boundary. Then there is no smooth map $\varphi : M \to \partial M$ which restricts to the identity on the boundary.

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$$\int_{\partial M} d(\partial M) = \int_{\partial M} \varphi^* d(\partial M)$$

$\varphi|_{\partial M} = id_{\partial M}$

$$= \int_M d(\varphi^*(d(\partial M))$$

$$= \int_M \varphi^* d(d(\partial M)) = 0.$$

Now the volume form defines a positive measure on ∂M and hence the first integral is positive. Contradiction. $\square$

Codifferential and Laplacian

Let (M, g) be an oriented Riemannian manifolds. The **Hodge-$*$-operator**

$$* : \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$$

is defined by the Hodge-$*$-operator on each $\Lambda^*(T_pM)$.

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The **Laplaceoperator** is the differentia~~l~~l operator

$$\Delta : \Omega^k(M) \longrightarrow \Omega^k(M)$$
$$\Delta := \underline{d\delta} + \delta d$$

with $\delta = 0$ on $\Omega^0(M)$ and $d = 0$ on $\Omega^n(M)$.

The Laplace–Beltrami Operator

$g = (g_{ij})$ Riemann tensor in oriented coordinates, g^{ij} its inverse, f smooth function. In coordinates we obtain $\Delta f = \delta(df)$

$$\Delta_g f = -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial f}{\partial x_j}\right)$$

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$$\int_M \langle \Delta\alpha, \alpha \rangle \, dM \geq 0$$

$M = \mathbb{R}^n$, $g = \langle ., . \rangle$

$$\Delta\Big(\sum_{\underline{I=\{1\leq i_1}<\ldots<i_k\leq n\}} \underline{\alpha_I} dx^{i_1} \wedge \ldots \wedge dx^{i_k}\Big) = \sum_{I=\{1\leq i_1<\ldots<i_k\leq n\}} (\Delta\alpha_I) \underline{dx^{i_1} \wedge} \ldots \wedge dx^{i_k}$$

where $\underline{\Delta\alpha_I}$ is the classical Laplace-operator on functions on $\mathbb{R}^n$.

$$\Delta\alpha_I = -\sum_{k=1}^{n} \frac{\partial^2 \alpha_I}{\partial x_k} \longleftarrow$$

Gauss' Divergence Theorem

Definition 37: Let X be a smooth vector field on an oriented Riemannian manifold (M, g). The **divergence** of X is the smooth real function $\mathrm{div}X$ defined by

$$\mathrm{div}X := \delta\big(\underbrace{g(X, .)}_{\in \Omega^1(M)}\big)$$

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Theorem 38: With the notation above

$$\int_M \mathrm{div}X \, dM = \overset{?}{-} \int_{\partial M} g(\underline{X, \mathbf{n}})d(\partial M).$$

where $\mathbf{n}$ is the outward normal along ∂M.

Gauss' Divergence Theorem

Proof: We have

$$\operatorname{div}X\,dM \underset{*1 = dM}{=} *\delta(\underbrace{g(X,.)}_{\in\Omega^1(M)}) = *(\overset{(-1)^{n(1-1)-1}}{-} * \underbrace{d(*g(X,.))}_{\in\Omega^n(M)})) \overset{**|_{\Omega^n(M)} = (-1)^{n(n-n)} = 1}{=} -d(*g(X,.)).$$

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$$\mathrm{div} X\, dM = *\delta(g(X,.)) = *(-*d(*g(X,.))) = -d(*g(X,.)).$$

Applying Stokes' Theorem we get

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It suffices to show

$$\iota_{\partial M}^*(*g(X,.)) = g(X,\mathbf{n}) d(\partial M).$$

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It suffices to show

$$\iota^*_{\partial M}(*g(X,.)) = g(X,\mathbf{n})\underline{d(\partial M)}.$$

Let $\{e_1, ..., e_{n-1}\}$ be an oriented orthonormal basis of $T_p(\partial M)$. Now with $e_0 := \mathbf{n}_p$ and $\underline{X_k := g_p(X, e_k)}$ and evaluating the left side we obtain

$$(*g(X,.))(e_1,...,e_{n-1}) = \Big(\underline{\sum_{k=0}^{n-1}} (-1)^k X_k e^0 \wedge \ldots \wedge \widehat{e^k} \wedge \ldots \wedge e^{n-1}\Big)(e_1,...,e_{n-1}) = X_0$$

$= X_0 = g(X, e_0) = g(X, n)$

which is equal to the right hand side. $\square$

<u>Proposition:</u> M compact, oriented, $\partial M = \emptyset$, g Riemannian metric.

$$\alpha \in \Omega^k(M), \quad \beta \in \Omega^{k+1}(M), \quad \gamma \in \Omega^k(M)$$

$$(d\alpha, \beta)_{L^2} = \int_M \langle d\alpha, \beta \rangle \, dM = \int_M \langle \alpha, \delta\beta \rangle \, dM = (\alpha, \delta\beta)_{L^2}$$

δ is the formal L^2-adjoint of $d : \Omega^k(M) \to \Omega^{k+1}(M)$

$$\Rightarrow \quad \int_M \langle \Delta\alpha, \gamma \rangle \, dM = \int_M \langle \alpha, \Delta\gamma \rangle \, dM$$

... Green's formula

Δ is a symmetric differential operator.