Differential Geometry II

Fibre Bundles

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Recall the definition of a *topological space*.

Set X, $\mathcal{O} \subset \mathcal{P}(X)$ is called topological structure:

1) $X, \emptyset \in \mathcal{O}$

2) $(U_i)_{i \in I} \subset \mathcal{O}$ family, then $\bigcup_{i \in I} U_i \in \mathcal{O}$

3) $U_1, \ldots, U_k \in \mathcal{O}$, then $\bigcap_{j=1}^{k} U_j \in \mathcal{O}$

$(X, \mathcal{O})$ topological space, $U \in \mathcal{O}$ is called open subset.

Ex: (X, d) metric space.

$$\mathcal{O}_{(X,d)} := \left\{ U \subset X \;\middle|\; \begin{array}{l} \forall p \in U \ \exists r > 0 : \\ B(p, r) \subset U \end{array} \right\}$$

Continuity: preimages of open subsets are open!

Fibre Bundles

Definition 39: A **fibre bundle** (E, B, π, F) of topological spaces consists of a continuous map $\pi : E \to B$ such that there for each point $p \in B$ there is an open neighbourhood $U \subset B$ and a homeomorphism

$$\Phi : \pi^{-1}(U) \to U \times F$$

so that $\mathrm{pr}_U(\Phi(e)) = \pi(e)$.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & \arrow[d, phantom, "\circlearrowright"] & U \times F \arrow[dl, "pr_U"] \\ & U & \end{tikzcd}$$

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Remark: It follows that each **fibre**, $\pi^{-1}(p) \cong F$ is homeomorphic to F.

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Lemma 40: Let $\pi : E \to B$ be a topological fibre bundle over a separable metric space B with fibre (homeomorphic to) a metric space F. Then E is a metrizable space.

~~The proof is left as an exercise.~~ See Proof of Lemma 43

Fibre bundles

Example: (1) The product $B \times F$ is called the **trivial** F–bundle over B. $\pi = pr_B$, $\mathcal{U} = B$ $\underline{\Phi} : B \times F \to B \times F$

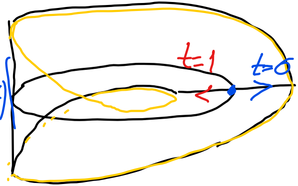
" id.

Fibre bundles

Example: (1) The product $B \times F$ is called the **trivial** F–bundle over B.
(2) The compact Moebius strip M^2 is a non-trivial $[-1,1]$–bundle over S^1.

$$M^2 \subset \mathbb{R}^3$$

$$= \{ (2\cos(2\pi t) + s\cos(\pi t),\ 2\sin(2\pi t),\ s\sin(\pi t)),\ t \in [0,1],\ s \in [-1,1] \} \subset \mathbb{R}^3$$



$$\downarrow \pi$$

$$S^1$$

$$\pi(2\cos(2\pi t) + s\cos(\pi t), \dots) = (\cos(2\pi t), \sin(2\pi t))$$

Fibre bundles

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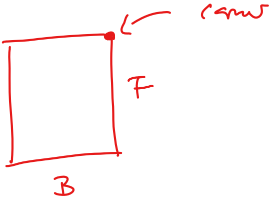
(2) The compact Moebius strip M^2 is a non-trivial $[-1, 1]$–bundle over S^1.
(3) If F is equipped with the discrete topology then (E, B, π, F) is also called a **covering (space)** of B. E.g. ∂M^2 of (2) is a (non-trivial) covering of S^1.

Fibre Bundles of Manifolds

Definition 41: A **fibre bundle of manifolds** is a fibre bundle (E, B, π, F) where E, B, F are manifolds, $\pi : E \to B$ is smooth and the local trivializations $\Phi : \pi^{-1}(U) \to U \times F$ can be chosen to be diffeomorphisms.

$\partial B = \emptyset$ or $\partial F = \emptyset$ else



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Remark: The projection π of a fibre bundle of manifolds is always a ~~surjection~~ submersion, i.e. for every $e \in E$ its differential $d_e\pi : T_eE \to T_{\pi(e)}B$ is surjective.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "pr_U"] \\ & U & \end{tikzcd} \quad \text{diffeo}$$

$$d_e\pi = \underbrace{dpr_U}_{= pr_{T_{\pi(e)}U}} \circ \underline{d\Phi} \quad \text{ism.}$$

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Examples: (1) $(B \times F, B, \mathrm{pr}_B, F)$ is the trivial bundle of manifolds.

(2) $(\mathbb{R}, \mathbb{R}, \pi, \{*\})$, where $\pi(x) = x^3$ is topological bundle but *not* a bundle of manifolds.

The Hopf Bundle

Examples: (3) Consider the 3–sphere as

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}.$$

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Recall that $S^2 \cong \mathbb{C}P^1$ are diffeomorphic where the complex projective line is defined as

$$\{[z_1, z_2] | (z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}\}$$

where $[z_1, z_2]$ denotes the equivalence class of (z_1, z_2) for the relation

$$(z_1, z_2) \sim \lambda(z_1, z_2)$$

for any $\lambda \in \mathbb{C} \setminus \{0\}$.

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$$(z_1, z_2) \sim \lambda(z_1, z_2)$$

for any $\lambda \in \mathbb{C} \setminus \{0\}$. Then $\pi : S^3 \to S^2$

$$\pi(z_1, z_2) := [z_1, z_2]$$

S^1.

is a fibre bundle of manifolds with fibre $\cancel{\pi}$. (Exercise)

Bundle Morphisms

$i=1,2$

Definition 42: Let (E_i, B_i, π_i, F_i), ~~$i_1, 2$~~ be two fibre bundles (of manifolds), $\varphi : B_1 \to B_2$ be a continuous map of their bases.

(smooth)

Bundle Morphisms

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$$\pi_2 \circ \Phi = \varphi \circ \pi_1.$$

$$\begin{tikzcd} E_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\circlearrowright"] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "\varphi"'] & B_2 \end{tikzcd}$$

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An **bundle isomorphism** is a bundle morphism $\Phi : E_1 \to E_2$ which is a homeomorphism (diffeomorphism). Notice, that its inverse is a bundle morphism and the covered map of the bases is also a homeomorphism (diffeomorphism).

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Example: A trivialization $\Phi : \pi^{-1}(U) \to U \times F$ is a isomorphism of $(\pi^{-1}(U), U, \pi, F)$ and $(U \times F, U, \mathrm{pr}_U, F)$.

Transition Functions

Let (E, B, π, F) be a topological fibre bundle.

$U, V \subset B$ open sets, $\Phi : \pi^{-1}(U) \to U \times F$ and $\Psi : \pi^{-1}(V) \to V \times F$ trivializations.

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$\Phi|_{\pi^{-1}(U \cap V)} : \pi^{-1}(U \cap V) \to (U \cap V) \times F$

homeomorphism

On $U \cap V$ the composition

$$\Psi \circ \Phi^{-1} : (U \cap V) \times F \to (U \cap V) \times F$$

has the form

$$\Psi \circ \Phi^{-1}(p, f) = (p, g(p, f))$$

where $g(p, .) : F \to F$ is a continuous family of homeomorphisms.
$g : U \cap V \to \mathrm{Homeo}(F)$ is called **transition function**.

Cocycle Condition

Lemma 43: (1) Let $\Phi_i : U_i \times F \to \pi^{-1}(U_i)$ $i = 1, 2, 3$ be three trivializations of the bundle over open subsets U_i and denote by $g_{ij} : U_i \cap U_j \to \mathrm{Homeo}(F)$ the transition function defined by $\Phi_i \circ \Phi_j^{-1}(p, f) = (p, g_{ij}(p, f))$.

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$$g_{ii}(p, .) = \text{id}_F \quad \forall p \in U_i$$

$$g_{ij}(p, .) \circ g_{jk}(p, , .) \circ g_{ik}(p, .) = \text{id}_F \quad \forall p \in U_1 \cap U_2 \cap U_3.$$

$k = i$

$\Longrightarrow \quad g_{ij} \circ g_{ji} = id_F \quad \forall\, p \in U_i \cap U_j$

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$$g_{ii}(p, .) = \mathrm{id}_F \quad \forall p \in U_i$$
$$g_{ij}(p, .) \circ g_{jk}(p, , .) \circ g_{ik}(p, .) = \mathrm{id}_F \quad \forall p \in U_i \cap U_j \cap U_k.$$

(2) Let $(U_i)_{i \in I}$ be an open covering of the topological space B and $\{g_{ij} : U_i \cap U_j \to \mathrm{Homeo}(F)\}_{i,j \in I}$ be a family of maps to the homeomorphism group of a topological space F which satisfy the condition of (1).

$(p, f) \in (U_i \cap U_j) \times F \mapsto g_{ij}(p, f) \in F$

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$$
\begin{aligned}
g_{ii}(p, .) &= \mathrm{id}_F \quad \forall p \in U_i \\
g_{ij}(p, .) \circ g_{jk}(p, , .) \circ g_{ik}(p, .) &= \mathrm{id}_F \quad \forall p \in U_{\cancel{1}\,i} \cap U_{\cancel{2}\,j} \cap U_{\cancel{3}\,k}.
\end{aligned}
$$

(2) Let $(U_i)_{i \in I}$ be an open covering of the topological space B and $\{g_{ij} : U_i \cap U_j \to \mathrm{Homeo}(F)\}_{i,j \in I}$ be a family of maps to the homeomorphism group of a topological space F which satisfy the condition of (1). Then there exists a fibre bundle (E, B, π, F) which admits trivializations over U_i whose transition functions are exactly given by $\{g_{ij}\}_{ij \in I}$. Any two such bundles are isomorphic.

$E := \coprod_{i \in I} (U_i \times F) \Big/ \sim$

$(x, f) \sim (x', f') \iff (x, f) = (x', f')$

or $x \in U_i,\ x' \in U_j,\ x = x'$ in B, $f' = g_{ij}(x, f)$

Cocycle Condition

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(3) Everything remains valid if we consider in (1) and (2) fibre bundles of manifolds and replace continuous by smooth, homeomorphisms by diffeomorphism.

Proof of Lemma 43

Same remarks as Lemma 40: Let $(U_i)_{i\in I}$ be open covering of E & $\Phi_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times F$ local trivializations.

Fact: Every metric space is paracompact: For every open covering of B there is a locally finite refinement $(V_j)_{j\in J}$: ($\forall\, p \in B$ $\exists$ open nbhd. $W \subset B$: $\#\{j \mid V_j \cap W \neq \emptyset\} < \infty$

We will construct such refinement for manifolds (using local compactness) assuming separability (leading to 2nd countability axiom of manifolds are assumed metrizable)

The proof of the more general fact uses axiom of choice and ordinals

Proof of Lemma 43

we define $\Phi_k : \pi^{-1}(V_k) \to V_k \times F$ by $\Phi_k := \Phi_\iota|_{\pi^{-1}(V_k)}$
for $\iota \in I : V_k \subset U_\iota$

$\rightsquigarrow$ metric on E: $e_1, e_2 \in E$

$$\tilde{d}(e_1,e_2) = \min \left\{ d_{V_k \times F}(\Phi_k(e_1), \Phi_k(e_2)) \;\middle|\; k \in K : e_1, e_2 \in \pi^{-1}(V_k) \right\}$$

$$d_{V_k \times F}((p,f),(p',f')) = d_B(p,p') + d_F(f,f') \quad \& \quad \min \emptyset = \infty$$

sym, deg., Δ-ineq., $\quad d(e_1,e_2) := \dfrac{\tilde{d}(e_1,e_2)}{1+\tilde{d}(e_1,e_2)}$

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Fibre Products

Given two fibre bundles (of manifolds) (E_i, B_i, π_i, F_i) the cartesian product $(E_1 \times E_2, B_1 \times B_2, \pi_1 \times \pi_2, F_1 \times F_2)$ where

$$\pi_1 \times \pi_2(e_1, e_2) := (\pi_1(e_1), \pi_2(e_2))$$

is a fibre bundle (of manifolds).

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is a fibre bundle (of manifolds).

If $C \subset B$ is a subset (submanifold), then the restriction of the bundle to C, $(\underline{E|_C} := \pi^{-1}(C), C, \pi|_{(E|_C)}, F)$ is a fibre bundle (of manifolds) over C.

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If $C \subset B$ is a subset (submanifold), then the restriction of the bundle to C, $(E|_C := \pi^{-1}(C), C, \pi|_{E|_C}, F)$ is a fibre (of manifolds) over C.

Definition 44: Given two fibre bundles (of manifolds) over the same base $(E_i, \underline{B}, \pi_i, F_i)$, their **fibre product** is defined by the restriction of their cartesian product to the diagonal

$$\Delta_B := \{(b, b) | b \in B\} \cong B$$

naturally identified with B by $(b, b) \mapsto b$.

Vector Bundles

Definition 45: A (real) **vector bundle** of rank k is a fibre bundle $(E, B, \pi, \mathbb{R}^k)$ (of manifolds), such that each fibre is a vector space and the trivializations can be chosen to be $\mathbb{R}$–linear on each fibre.

$$\Phi : \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times \mathbb{R}^k$$

$\lambda \in \mathbb{R}$, $q \in \mathcal{U}$, $v, w \in \pi^{-1}(q) \mapsto v + w$, λv

$$\Phi(v) = (q, \varphi_q(v)) \quad \forall v \in \pi^{-1}(q)$$

$$\Phi(v+w) = (q, \varphi_q(v+w))$$

$$= (q, \varphi_q(v) + \varphi(w))$$

$\varphi_q : \pi^{-1}(q) \to \mathbb{R}^k$ homomorphism of vector spaces

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Definition 45: A (real) **vector bundle** of rank k is a fibre bundle $(E, B, \pi, \mathbb{R}^k)$ (of manifolds), such that each fibre is a vector space and the trivializations can be chosen to be $\mathbb{R}$–linear on each fibre. In particular, the transition function for two such trivializations is a continuous (smooth) map

$$\begin{array}{c} \mathrm{Homeo}(\mathbb{R}^k) \\ \cup \end{array}$$

$$g : U \cap V \longrightarrow Gl(k; \mathbb{R}).$$

Remark: For two vector bundles over the same base $i = 1, 2$ $(E_i, B, \pi_i, \mathbb{R}^{k_i})$ denote by $E_1 \oplus E_2$ their fibre product. Then the addition defines a morphism of fibre bundles (of manifolds)

$$E \oplus E \to E$$

as well as the scalar multiplication

$$\underline{\mathbb{R}} \times E \to E.$$

Vector Bundles

Morphisms of vector bundles are morphisms of bundles (of manifolds) which restricts to each fibre as a homomorphism of vector ~~bundles~~. spaces

Examples: (1) The trivial vector bundle $B \times \mathbb{R}^k$.

(2) The Moebius bundle:

$$E \subset S^1 \times \mathbb{R}^2$$

$$= \left\{ (\cos(2\pi t), \sin(2\pi t), s\cos(\pi t), s\sin(\pi t)) \;\middle|\; \begin{array}{l} t \in [0,1] \\ s \in \mathbb{R} \end{array} \right\}$$

$E_1 \subset E$ closed subset

$$= \{ (\dots) \mid s \in [-1,1] \}$$

$E_1 \cong M$ ... Moebius strip

diffeo – fibre bdl. isomorphism.

Tangent Bundles