Differential Geometry II

Vector Bundles

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Cocycle Condition

$\pi^{-1}(U_i) \longrightarrow U_i \times F$

Lemma 43: (1) Let $\Phi_i : \cancel{U_i \times F \to \pi^{-1}(U_i)}$ $i = 1, 2, 3$ be three trivializations of the bundle over open subsets U_i and denote by $g_{ij} : U_i \cap U_j \to \mathrm{Homeo}(F)$ the transition function defined by $\Phi_i \circ \Phi_j^{-1}(p, f) = (p, \underline{g_{ij}(p, f)})$.

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$$\begin{aligned} g_{ii}(p,.) &= \mathrm{id}_F \quad \forall p \in U_i \\ g_{ij}(p,.) \circ g_{jk}(p,,.) \circ g_{\underset{ki}{\cancel{ik}}}(p,.) &= \mathrm{id}_F \quad \forall p \in U_1 \cap U_2 \cap U_3. \end{aligned}$$

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(2) Let $(U_i)_{i \in I}$ be an open covering of the topological space B and $\{g_{ij} : U_i \cap U_j \to \text{Homeo}(F)\}_{i,j \in I}$ be a family of maps to the homeomorphism group of a topological space F which satisfy the condition of (1).

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(2) Let $(U_i)_{i \in I}$ be an open covering of the topological space B and $\{g_{ij} : U_i \cap U_j \to \text{Homeo}(F)\}_{i,j \in I}$ be a family of maps to the homeomorphism group of a topological space F which satisfy the condition of (1). Then there exists a fibre bundle (E, B, π, F) which admits trivializations over U_i whose transition functions are exactly given by $\{g_{ij}\}_{ij \in I}$. Any two such bundles are isomorphic.

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(3) Everything remains valid if we consider in (1) and (2) fibre bundles of manifolds and replace continuous by smooth, homeomorphisms by diffeomorphism.

Proof of Lemma 43

$$E := \coprod_{i \in I} (U_i \times F) \Big/ \sim \quad \xrightarrow{\pi} B$$

$$[(x,f)_i] \longmapsto x$$

$\sim$: $\forall i,j \in I \; \forall x \in U_i \cap U_j, \forall f \in F$:

$$(x,f)_j \sim (x, g_{ij}(x,f))_i$$

Quotient topology : X top. space, $\sim$ equivalence rel. on X

quotient topology on $X/\sim$: finest topology s.t.

$$p : X \to X/\sim \qquad p(x) := [x] \text{ is continuous.}$$

in short: $U \subset X/\sim$ open $\iff$ $p^{-1}(U) \subset X$ is open.

$U \subset B$ open $\qquad p^{-1}(\pi^{-1}(U)) = \coprod_{i \in I} (U \cap U_i) \times F$ open!

$\Rightarrow \pi^{-1}(U) \subset E$ is open $\Rightarrow \pi$ is continuous.

Proof of Lemma 43

$$\pi^{-1}(U_i) = p(U_i \times F) \subset E \quad \text{open}$$

$$p|_{U_i \times F} : U_i \times F \to p(U_i \times F) \subset E \quad \text{bijection!}$$

$$(p|_{U_i \times F})^{-1} : \pi^{-1}(U_i) \to U_i \times F \quad \text{continuous.}$$

$$W \subset U_i \times F \quad \text{open}$$

need to show $\quad p(W) \subset E$ is open

$$\Leftrightarrow \; p^{-1}(p(W)) = \coprod_{j \in I} \underbrace{\Phi_{ij}^{-1}(W)}_{\text{open}} \quad \text{open.}$$

$$\Phi_{ij}(x, f) = (x, g_{ij}(x, f))$$

Proof of Lemma 43

<u>At uniqueness</u>: Let $\tilde{E} \xrightarrow{\tilde{\pi}} B$ be another bundle with fibre F & local trivializations

$$\tilde{\Phi}_i : \tilde{\pi}^{-1}(U_i) \to U_i \times F \quad \text{s.t.}$$

$$\to \quad \tilde{\Phi}_i \circ \tilde{\Phi}_j^{-1}(x, f) = (x, \underline{g_{ij}}(x, f))$$

Define an isomorphism $\psi : \tilde{E} \to E$

$$\begin{tikzcd}
\tilde{\pi}^{-1}(U_i) \arrow[r, "\tilde{\Phi}_i"] \arrow[d, "\psi|_{\tilde{\pi}^{-1}(U_i)}", "\cong"'] & U_i \times F \arrow[dl, "\Phi_j^{-1}"] \\
\pi^{-1}(U_i) &
\end{tikzcd}$$

$\square$

Vector Bundles

Definition 45: A (real) **vector bundle** of rank k is a fibre bundle $(E, B, \pi, \mathbb{R}^k)$ (of manifolds), such that each fibre is a vector space and the trivializations can be chosen to be $\mathbb{R}$–linear on each fibre.

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$$g : U \cap V \longrightarrow Gl(k; \mathbb{R}).$$

$E_p := \pi^{-1}(b)$

Remark: For two vector bundles over the same base $(E_i, B, \pi_i, \mathbb{R}^{k_i})$ denote by $E_1 \oplus E_2$ their fibre product. Then the addition defines a morphism of fibre bundles (of manifolds)

$$E \oplus E \to E$$

as well as the scalar multiplication

$$\mathbb{R} \times E \to E.$$

$E \xrightarrow{\pi} B$ vector bdle.

$0_E \subset E$

$= \{ 0 \in E_p \mid p \in B \} \overset{\pi}{\cong} B$

The Tangent Bundle

Proposition 46: (1) The disjoint union of the family $\{T_pM\}_{p\in M}$ of tangent spaces of a smooth manifold of dimension n forms a smooth vectorbundle of rank n over that manifold where the projection is given by

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(2) Given a chart (U, φ, V) the map

$$X \in \pi^{-1}(U) \mapsto (\pi(X), (\cancel{X_1}, \ldots, \cancel{X_n})) \in U \times \mathbb{R}^n$$

where

$$X = \sum_{j=1}^{n} X_j \frac{\partial}{\partial x_j}(\pi(X)),$$

is a trivialization of this bundle.

local

The Tangent Bundle

Proof: Let $\boxed{\phi} := \tilde{\varphi}^{-1} \circ \varphi : \varphi^{-1}(U \cap \tilde{U}) \to \tilde{\varphi}^{-1}(U \cap \tilde{U})$ be the transition map between two charts from the differentiable atlas $\underline{(U, \varphi, V)}$ and $\underline{(\tilde{U}, \tilde{\varphi}, \tilde{V})}$.

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$$\frac{\partial}{\partial \tilde{x}_i} = \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x_j} \frac{\partial}{\partial x_j}.$$

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Hence the transition function of the prospective trivializations are given by $g : U \cap \tilde{U} \to Gl(n; R)$

$$g(p) = \left(\frac{\partial \phi_i}{\partial x_j} \right)_{i,j=1}^{n} = D_{\varphi^{-1}(p)} \phi$$

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M was a smooth manifold, the transition map ϕ is smooth, hence g is smooth.

The Tangent Bundle

$\Phi : V \times \mathbb{R}^n \to \pi^{-1}(U)$ given by

$$\Phi(x, v) := \Big(\varphi(x), \sum_{j=1}^{n} v_j \frac{\partial}{\partial x_j}\Big)$$

defines a bijective map such that the transition maps between any two of them are differentiable. Therefore they form a differentiable atlas of the **tangent space**

$$TM := \coprod_{p \in M} T_p M.$$

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Hence TM is a manifold.

Alternatively: equip M with a Riemannian metric g (partition of unity required!) and define length $l(\gamma)$ for $\gamma : (0,1] \to TM$ differentiable

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$\in T_{\varphi(x)}M$

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Hence TM is a manifold. The projection map $\pi : TM \to M$ w.r.t. any of the charts $(\pi^{-1}(U), \Phi, V \times \mathbb{R}^n)$ takes the form

(U, φ, V)

$$\varphi^{-1} \circ \underbrace{\pi \circ \Phi(x, v)}_{= \varphi(x)} = x$$

and is hence smooth.

Fibres, Sections and Vector Fields

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A smooth **section** is smooth map $\sigma : B \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$. The image $\sigma(B) \subset E$ of a smooth section is a submanifold.

$$\begin{tikzcd} E \arrow[d, "\pi"'] & \\ B \arrow[r, "\mathrm{id}"'] & B \arrow[ul, "\sigma"'] \end{tikzcd}$$

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Example: Consider $T(TM)$ – the tangent space of the tangent space of a manifold. For each $v \in TM$ there is a canonical subspace $T_{\pi(v)}M \cong T_v(T_{\pi(v)}M) \subset T_v(TM)$ – the tangent space to the fibre $T_{\pi(v)}M$.

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Then $X : TM \to T(TM)$ defined by

$$X(v) := v$$

is a smooth vector field on TM, called the **Euler field**.

The Cotangent Bundle

$(T_pM)^*$

$\|$

Similarly, the cotangent spaces $\{T_p^*M\}_{p\in M}$ form the **cotangent bundle**, T^*M, of M

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$\pi^*: T^*M \to M$ is given by

$$\pi^*(\alpha) = p \text{ for } \alpha \in T_p^*M.$$

With a differentiable chart (U, φ, V)

$$\alpha \in (\pi^*)^{-1}(U) \mapsto (\pi(\alpha), (\alpha(\frac{\partial}{\partial x_1}), \ldots, \alpha(\frac{\partial}{\partial x_n}))) \quad \in T^*M$$

provides the local trivializations.

The Cotangent Bundle

With $\phi = \tilde{\varphi}^{-1} \circ \varphi$ for differentiable charts as before the transition map $g^* : U \cap \tilde{U} \to Gl(n;\mathbb{R})$ from the trivialization related to (U, φ, V) to the trivialization related to $(\tilde{U}, \tilde{\varphi}, \tilde{V})$ is given by

$$g^*(p) = (g(p)^{-1})^T.$$

The **tautological one form**, $\theta \in \Omega^1(T^*M)$ is defined by

$$\theta_\alpha(X) := \alpha(\cancel{d_p(X)}) \qquad \alpha(d_\alpha \pi^*(X)) \in \mathbb{R}$$

where $\alpha \in T_p^*M$ and $X \in T_\alpha(T^*M)$.

$$\pi^* : T^*M \to M$$

$$d_\alpha \pi^* : T_\alpha(T^*M) \to T_{\pi(\alpha)}M = T_pM$$

$$\begin{array}{ccc} \in & & \in \\ X & \longmapsto & d_\alpha \pi^*(X) \end{array}$$

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Exercise: Express θ in coordinates of T^*M around α provided by a chart of M and compute its exterior derivative $d\theta$.

Subbundles

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Definition 47: Let $E \xrightarrow{\pi} M$ be a smooth vector bundle.
(1) A **subbundle of** E is a submanifold $F \subset E$ such that $\pi(F) = M$ and for any $p \in M$, $\underbrace{\pi^{-1}(p)}_{= E_p} \cap F \subset E_p$ is a linear subspace.
(2) The **dual (bundle)**, $E^* \xrightarrow{\pi^*} M$ of E is given by

$$E^* := \coprod_{p \in M} (E_p)^*$$

and the obvious projection map π^* together with the following trivializations: Let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ be a local trivialization of E then $(\Phi^{-1})^* : (\pi^*)^{-1}(U) \to U \times (\mathbb{R}^k)^*$ assigns

$$\alpha \in \overset{E_p^*}{\cancel{T_p^* M}} \mapsto \alpha \circ \Phi(p, .)^{-1} \in (\mathbb{R}^k)^*$$

$\Phi(p,.)^{-1} : \mathbb{R}^n \longrightarrow E_p$ linear

Tensor Products

Definition 48: Given to vector bundles $E_i \xrightarrow{\pi_i} M$ their **tensor product** $E_1 \otimes E_2$ ~~to~~ $\xrightarrow{\pi}$ M is given by

$$E_1 \otimes E_2 = \coprod_{p \in M} B((E_1)^*_p, (E_2)^*_p)$$

where $B((E_1)^*_p, (E_2)^*_p)$ denotes the vector space of bilinear forms $\alpha : (E_1)^*_p \times (E_2)^*_p \to \mathbb{R}$. or maps

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Remark: This definition is only good if the rank of the bundles is finite! Then $(E_i)^{**}_p \cong (E_i)_p$. Notice that the fibre of $E_1^* \otimes E_2^*$ is given by bilinear maps $B(E_1, E_2)$.

for vector spaces

Exercise: $(E_1 \otimes E_2) \otimes E_3 = E_1 \otimes (E_2 \otimes E_3) = \mathrm{Mult}(E_1^* \otimes E_2^* \otimes E_3^*)$