Homework Set 1

Floer Homology 2019

Discussion in Tutorials

Problem 1
Show the conservation of energy in an autonomous Hamiltonian system, i.e. for any flowline of the Hamiltonian vector field X_H of (M, ω, H) as explained in class, the value of H is constant.

Problem 2
This series of questions aims to understand, that for a family of diffeomorphisms $\{\Phi_t\}$ associated to the family of Hamiltonian vector fields X_{H_t}, e.g. the flow of X_H if H is autonomous, the symplectic structure is preserved.
(a) Let X be a vector field and Φ_t be a family of diffeomorphisms ($t \in (-\epsilon, \epsilon)$) with $\Phi_0 = id$ such that

$$\frac{d}{dt}\Big|_{t=0} \Phi_t = X.$$

Then we define the Lie-derivative of a k–form α along X via

$$\mathcal{L}_X := \frac{d}{dt} \Phi_t^* \alpha.$$

Show that this definition is independent of the choice of Φ_t. Explain, that it is possible to find such a family at least in a neighbourhood of each point.
(b) Let Φ_t be a family of diffeomorphisms. Define the family of vectorfields X_t via

$$X_t := (\frac{d}{dt} \Phi_t) \circ \Phi_t^{-1}.$$

Explain why for a k-form α we have

$$\frac{d}{dt} \Phi_t^* \alpha = \Phi_t^* (\mathcal{L}_{X_t} \alpha).$$

(c) Show for a vectorfield X and differential forms α and β that $\mathcal{L}_X d\alpha = d(\mathcal{L}_X \alpha$ and $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X \alpha \wedge \beta + \alpha \wedge \mathcal{L}_X \beta$.
(d) Show for a differential form α and vectorfield X that

$$\mathcal{L}_X \alpha = i_X d\alpha + d(i_X \alpha)$$

where $i_X \alpha$ denotes the insertion of X into the first position of the multi-linear forms α and $d\alpha$.
Hint: Show that the assertion is true for funtions. Show that any k–form can be written in a neighbourhood of a point as a sum of terms of the form $df \wedge \beta$ where f is a function and β a $(k-1)$–form. Then use the previous exercises.
(e) Finally, derive the assertion on Hamiltonian diffeomorphisms mentioned at the beginning.

Problem 3
(a) Study Moser's trick (any modern text book, e.g. Ana Cannas da Silva's book or the internet).
(b) Let ω_t be a family of symplectic forms which are cohomologous. Explain how it can be approximated by a piecewise smooth (in t) differentiable family ω'_t such that $\omega'_t - \omega_{t_0} = d\alpha_t$ for a piecewise smooth family of one forms α_t.
(c) Show that under appropriate assumptions all members of a family of symplectic structures given in (b) are symplectomorphic.
(d) Prove Darboux's Theorem (find it in literature if necessary).
(e) Formulate Weinstein's symplectic neighborhood theorem (see literature) and understand how to prove it.