Homework Set 2

Floer Homology 2019

Discussion in Tutorials

Problem 1
Fundamental Lemma of variational calculus: Let $u, v : \mathbb{R} \to \mathbb{R}$ be two continuous functions, which satisfy the following condition: For any differentiable function φ with compact support in the open intervall (a, b)

$$\int_a^b \varphi(t)u(t) + \varphi'(t)v(t)dt = 0.$$

Then v is differentiable and $v' = u$.

Problem 2
Lagrangian Mechanics: Let $L : TM \times \mathbb{R} \to \mathbb{R}$ be a differentiable function, 1–periodic in the second component, TM is the tangent space of a manifold M. We study the extremal points of the following *Lagrangian functional*: To a differentiable closed (i.e. 1–periodic) curve $\gamma : \mathbb{R} \to M$ we assign

$$\mathcal{L}(\gamma) := \int_0^1 L(\gamma(t), \dot{\gamma}(t), t)dt.$$

(a) Derive the differential condition for critical points of $\mathcal{L}$, the so–called *Euler-Lagrange equations* in local coordinates. Hint: Use the result in Problem 1
(b) Express that condition more explicitely for the case $L(p, v) := g_p(v, v)$, for a Riemannian metric g on M, $p \in M$ and $v \in T_pM$, i.e. the Lagrangian functional is the length of the curve w.r.t. g.
$(c)^*$ Describe the Hessian for the length functional.

Problem 3
Hamiltonian Action Functional. In class we will discuss the following functional: Let (M, ω) be a symplectic manifold, $H : M \times \mathbb{R} \to \mathbb{R}$ a differentiable function, 1–periodic in the second component. A closed, 1–periodic curve $\gamma : \mathbb{R} \to M$ is called contractible if there is a differentiable map $u : D^2 \to M$ from the disc such that $u(e^{2\pi i t}) = \gamma(t)$ for all $t \in \mathbb{R}$. To a pair (γ, u) we assign its *Hamiltonian action*

$$\mathcal{A}(\gamma, u) := \int_{D^2} u^*\omega + \int_0^1 H(\gamma(t), t)dt.$$

Describe this funtional as a Lagrangian functional on the tangent bundle of the symplectic manifold. Start with the standard symplectic manifold. Discuss critical points.