Homework Set 3

Floer Homology 2019

Discussion in Tutorials

Problem 1
(a) Let (M, ω) be a closed symplectic manifold. Show that for all positive integers $k \leq \dim M/2$ $\omega^k = \underbrace{\omega \wedge ... \wedge \omega}_{k}$ is not exact. What does it mean for the homology of M?
(b) Which of the following closed manifolds cannot carry a symplectic structure: spheres, tori, Klein-bottle, $S^3 \times S^1$, $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$, $\mathbb{C}P^2 \sharp \mathbb{C}P^2$, $\mathbb{C}P^2 \sharp \mathbb{C}P^2 \sharp \mathbb{C}P^2$. $\sharp$ denotes the connected sum of manifolds, $\mathbb{C}P^2$ is the complex projective plain oriented as a complex manifold, $\overline{\mathbb{C}P^2}$ is the same space with the opposite orientation. Remark: Try to answer the question for es many spaces as possible. The hard cases are interesting and you will get a hint in class what kind of mathematics is needed to decide it.

Problem 2
(a) Discuss that $\widetilde{\Omega}_0(M)$ as defined in class is the universal cover of the space of contractible loops $\Omega_0(M)$. Explain why the fundamental group of the latter is $\pi_1(\Omega_0(M)) \cong \pi_2(M)$.
(b) Prove that the Hamiltonian action functional $\mathcal{A}_H : \widetilde{\Omega}_0(M) \to \mathbb{R}$ is well-defined.
$(c)^*$ Generalise $\mathcal{A}_H$ to the universal covers of all connected components of the space of loops in M, $\Omega(M)$.

Problem 3
Show the following identity from the lecture

$$\frac{d}{d\tau}\Big(\mathcal{A}_H(\gamma_\tau, u_\tau) - \int_{D^2} u_{-\epsilon}^* \omega\Big) = \frac{d}{d\tau}\Big(\int_{-\epsilon}^{\tau}\int_0^1 \omega(\frac{d}{ds}\gamma_s(t), \gamma_s'(t))dtds - \int_0^1 H(\gamma_\tau(t), z)dt\Big).$$

Problem 4
(a) Give examples of symplectically aspherical, closed symplectic manifolds.
(b) Explain why the action functional $\mathcal{A}_H$ descends to a functional on $\Omega_0(M)$ in the case of a symplectically aspherical manifold.
(c) Show that even in this case, $\mathcal{A}_H$ is unbounded from above and below. Hence minimising sequences cannot converge to a critical point in any reasonable way.

Problem 5
(a) Let $f : M \to \mathbb{R}$ be a differentiable function. Recall the definition of the Hessian of f in a critical point p. Explain that this is only well-defined since p is critical.
(b) Let f be a Morse function and X be a differentiable vector–field on M such that $X(f) \geq 0$ and equality holds exactly at the critical points of f (in the lecture this was called gradient-like, but for these it is usually asked for stronger conditions). Check the following hypothesis: X vanishes exactly at the critical points of f.

Problem 6
(a) Let $U, S \subset M$ be two differentiable submanifolds of M which intersect transversely. Show that the intersection $U \cap S$ is a differentiable submanifold of M of dimension

$$\dim U \cap S = \dim U + \dim S - \dim M.$$

Find a formula for the codimension, where $\text{codim} X = \dim M - \dim X$ for a submanifold of M.
(b) Show the identity from the lecture by providing the bijection

$$U_p \cap S_q = \{\gamma : \mathbb{R} \to M \mid \gamma'(t) = -X(\gamma(t)), \lim_{t\to\infty} \gamma(t) = q, \lim_{t\to-\infty} \gamma(t) = p\}.$$