Homework Set 5

Floer Homology 2019

Problem 1
Let ω be the volume form on the unit sphere $S^2 \subset \mathbb{R}^3$ and $H_a : S^2 \to \mathbb{R}$ be the function: $h(x, y, z) = az$.
(a) Determine the value of the Hamiltonian action for the circles with constant z–coordinate.
(b) Determine all critical points of the corresponding Hamiltonian action functional. (c) For which values of a are all critical points non-degenerate? Determine the Conley-Zehnder indices of the stationary critical points with the trivial spanning disc in this case.

Problem 2
Show that the following two families $A, B : [0, 1] \to M(2, \mathbb{R})$ are families of symplectic matrices which meet the conditions $A(0) = B(0) = \mathbb{E}$ and the endpoints do not have eigenvalue 1. Determine their Conley–Zehnder indices.
(a)

$$A(t) = \begin{pmatrix} 1 + 4\pi^2 t^2 & 2\pi t \\ 2\pi t & 1 \end{pmatrix}$$

(b)

$$B(t) = \begin{pmatrix} 1 - 4\pi^2 t^2 & -2\pi t \\ 2\pi t & 1 \end{pmatrix}$$

Remark: (b) is more subtle.

Problem 3
Show that for two paths $A, B : [0, 1] \to M(k, \mathbb{R})$ where $A(0) = B(0) = B(1) = \mathbb{E}$ the pointwise products $BA, AB : [0, 1] \to M(k, \mathbb{R})$, $AB(t) = A(t)B(t)$ of the two are both homotopic to the concatenation $B \cdot A : [0, 1] \to M(k, \mathbb{R})$

$$B \cdot A(t) := \begin{cases} B(2t) & \text{for } t \in [0, \frac{1}{2} \\ A(2t-1) & \text{for } t \in [\frac{1}{2}, 1] \end{cases} .$$

Problem 4
(a) Let J_0 be the standard complex structure on $\mathbb{R}^{2n}$ as introduced in class and S be symmetric $2n \times 2n$–matrix. Show that $\exp(J_0 S)$ is a symplectic matrix with respect to the symplectic form which in the standrd basis is given by J_0.
(b) More generally, show that for a 1–parameter family $S(t)$ of symmetric matrices a family $R(t)$ of matrices solving

$$R'(t) = J_0 S(t) R(t), \quad R(0) = \mathbb{E}$$

is symplectic. Vice versa, if $R(t)$ is a family of symplectic matrices starting at $\mathbb{E}$ and solving teh equation, then $S(t)$ is symmetric for all t.

Problem 5
Given a symplectic form ω on a vector space V and a euclidean scalar product g, by the condition

$$\omega(v, Aw) = g(v, w)$$

$A : V \to V$ is a uniquely defined endomorhism. A^2 is symmetric w.r.t. the scalar product and negative definite. Hence the square root of $-A^2$ is defined and $J := \sqrt{-A^2}^{-1} A$ is a compatible almost complex structure.