Homework Set 6

Floer Homology 2019

Problem 1
Let $\gamma : [0,1] \to \mathbb{C}$ be a differentiable loop (also at $t = 0, 1$) with Fourier expansion

$$\gamma(t) = \sum_{k=-\infty}^{\infty} a_k^0 e^{2k\pi i t}.$$

(a) We define the space of such functions

$$H^m := \{\gamma \mid \sum_{k=-\infty}^{\infty} k^{2m}|a_k|^2 < \infty\}$$

with the square root of the sum defining a norm, which turns that into a Banach space. If $m-1 \geq j$ then $C^m \hookrightarrow H^m \hookrightarrow C^j$ continously with respect to the norms (see e.g. Hofer/Zehnder: Symplectic Invariants and Hamiltonian Dynamics or try to prove that yourself).

If $u : I \times [0,1] \to \mathbb{C}$ is a solution of the Cauchy–Riemann–equation $\partial_s u + i\partial_t u = 0$ for an open interval I containing 0 then

$$u(s,t) = \sum_{k=-\infty}^{\infty} e^{2k\pi(s+it)}.$$

(b) For any $m \in \mathbb{N}$ find a function $\gamma \in H^m$ which does not admit such a solution u on any such interval.
(c) For any ϵ find a smooth function γ which does not admit such a solution u on the interval $I = (-\epsilon, \epsilon)$.
$(d)^*$ Find a smooth function γ which does not admit such a solution u on any such interval.

Problem 2
Let $u : \mathbb{R} \times [0,1] \to M$ be a solution of Floer's equation.
(a) Explain that the action functional is strictly decreasing along non-stationary u, i.e. $\mathcal{A}_{H_t}(u(s,.))$ strictly decreases as a function of s.
(b) For $u : [a,b] \times [0,1] \to M$ define the energy as

$$E(u) := \int_{I\times[0,1]} |\partial_s u|^2 + |\partial_t u - X_{H_t} \circ u|^2 ds dt.$$

With a spanning disc v_a for $u(a,.)$ fixed define a piecewise differentiable spanning disc v_b for $u(b,.)$ with bounded differential on the pieces. Show that if u satisfies Floer's equation then

$$E(u) = \mathcal{A}_{H_t}(u(a,.), v_a) - \mathcal{A}_{H_t}(u(b,.), v_b).$$

(c) Show the identity for the energy functional for a solution $u : \mathbb{R} \times [0,1] \to M$ of Floer's equation

$$E(u) = \mathcal{A}_{H_t}(\tilde{\alpha}) - \mathcal{A}_{H_t}(\tilde{\beta})$$

where $\lim_{s\to-\infty} u(s,.) = \alpha$, $\lim_{s\to\infty} u(s,.) = \beta$, $\alpha, \beta \in \mathrm{Crit}(\mathcal{A}_{H_t}$ and $\tilde{\alpha}, \tilde{\beta}$ denote the corresponding loops together with a choice of a homotopy class of a spanning disc, such that the union of the disc of $\tilde{\alpha}$, u and the disc of $\tilde{\beta}$ with the opposite orientation yields a map $v : S^2 \to M$ which is contractible.
(d) Show that the map $u : \mathbb{R} \times [0,1] \to S^2 = \mathbb{C} \cup \{\infty\}$ into the Riemann sphere goven by

$$u(s,t) = \exp(e^{2\pi(s+\mathrm{i}t)})$$

is 1–periodic in t and has unbounded energy: $E(u) = \infty$. Discuss the limits $\lim_{s\to\pm\infty} u(s,.)$.

Problem 3
Let (M,ω) be a symplectic manifold ($d\omega = 0$ is not necessary for the following). Show that the set k times differentiable of compatible almost complex structures $\mathcal{J}^k(M,\omega)$ is a Banach manifold. The topology is for instance given by the C^k-topology of matrix-valued functions on open subsets of $\mathbb{R}^{2n}$. The emphasis is hereby on the set being parametrized by open neighborhoods of a Banach space but you should also be aware that one has to check separability (a countable dense subset). One way of checking the former is to describe $\mathcal{J}^k(M,\omega)$ as sections of a bundle whose fibres are quotients of Lie groups. Start with a smooth compatible almost complex structure. How can all other elements of $\mathcal{J}^k(M,\omega)$ be described?

Problem 4
This exercise leads to the first part of the theorem on solutions of Floer's equation with finite energy (see lecture).
(a) Inform yourself on Arzelá's and Ascoli's theorem.
Let (M,ω) be a closed symplectic manifold, $H : M \times [0,1] \to \mathbb{R}$ a smooth 1–periodic function, $\{J\}_{t\in[0,1]}$ be a smooth 1–periodic family of almost complex structures compatible to ω. Let $u : \mathbb{R} \times [0,1] \to M$ be a smooth solution of Floer's equation with finite energy: $E(u) < \infty$.
(b) Explain that for any sequence $(s_k)_k$ which tends to either $\pm\infty$ there is a subsequence (s'_k) such that $u(s'_k, 0)$ converges in M and such that

$$\lim_{k\to\infty} \int_0^1 \|\partial_t u(s_k,t) - X_{H_t}(u(s_k,t))\|^2_{g_t} dt = 0.$$

The norm $\|.\|_{g_t}$ is given by the Riemannian metric g_t defined by ω and J_t.
(c) Show that the sequence in (b) satisfies Arcel's and Ascoli's theorem and formulate the consequence of it.
(d) Derive the following result: For any sequence $(s_k)_k$ which tends to either $\pm\infty$ there is a subsequence (s'_k) such that $u(s'_k, 0)$ converges in C^ℓ to a critical point of $\mathcal{A}_H$, i.e. to a periodic solution of Hamilton's equation. $\ell \in \mathbb{N}$ is not bigger than the degree of differentiability required for the data like J_t.