Homework Set 7

Floer Homology 2019

Problem 1
Let (M, ω) be a symplectic manifold ($d\omega = 0$ is not necessary for the following). Show that the set k times differentiable of compatible almost complex structures $\mathcal{J}^k(M, \omega)$ is a Banach manifold. The topology is for instance given by the C^k-topology of matrix-valued functions on open subsets of $\mathbb{R}^{2n}$. The emphasis is hereby on the set being parametrized by open neighborhoods of a Banach space but you should also be aware that one has to check separability (a countable dense subset). One way of checking the former is to describe $\mathcal{J}^k(M, \omega)$ as sections of a bundle whose fibres are quotients of Lie groups. Start with a smooth compatible almost complex structure. How can all other elements of $\mathcal{J}^k(M, \omega)$ be described?

Problem 2
This exercise leads to the first part of the theorem on solutions of Floer's equation with finite energy (see lecture).
(a) Inform yourself on Arzelá's and Ascoli's theorem.
Let (M, ω) be a closed symplectic manifold, $H : M \times [0, 1] \to \mathbb{R}$ a smooth 1–periodic function, $\{J\}_{t\in[0,1]}$ be a smooth 1–periodic family of almost complex structures compatible to ω. Let $u : \mathbb{R} \times [0, 1] \to M$ be a smooth solution of Floer's equation with finite energy: $E(u) < \infty$.
(b) Explain that for any sequence $(s_k)_k$ which tends to either $\pm\infty$ there is a subsequence (s'_k) such that $u(s'_k, 0)$ converges in M and such that

$$\lim_{k\to\infty} \int_0^1 \|\partial_t u(s_k, t) - X_{H_t}(u(s_k, t))\|^2_{g_t} dt = 0.$$

The norm $\|.\|_{g_t}$ is given by the Riemannian metric g_t defined by ω and J_t.
(c) Show that the sequence in (b) satisfies Arcel's and Ascoli's theorem and formulate the consequence of it.
(d) Derive the following result: For any sequence $(s_k)_k$ which tends to either $\pm\infty$ there is a subsequence (s'_k) such that $u(s'_k, 0)$ converges in C^ℓ to a critical point of $\mathcal{A}_H$, i.e. to a periodic solution of Hamilton's equation. $\ell \in \mathbb{N}$ is not bigger than the degree of differentiability required for the data like J_t.
(e) Given (M, ω) and time-dependent H such that all 1–periodic solution to Hamilton's equation are non-degenerate. Explain that they must be isolated in the loop space $\Omega(M)$ equipped with C^ℓ-topology.
(f) Under the assumption of (e) show that for any solution u of Floer's equation with $E(u) < \infty$ that there exist 1–periodic solutions α, β of Hamilton's equation such that $\lim_{s\to-\infty} u(s, .) = \alpha$ and $\lim_{s\to-\infty} u(s, .) = \beta$ in C^ℓ.

Problem 3
(a) Study the definition and properties of the first Chern class of a complex or symplectic vector bundle over a closed oriented surface. Extend that definition to smooth manifolds of higher dimension (see McDuff/Salamon: Introduction to Symplectic Topology, Section 2.6.
(b) Given to differentiable spanning discs of a non-degenerate 1–periodic solutions to Hamilton's equation. Show that the difference of their Conley-Zehnder indices is given by the twice the Chern number of the pull-back of the tangent bundle to the sphere obtained by gluing along the boundaries one disc to the other with opposite orientation.
(c)* Determine the Chern class of $\mathbb{C}P^n$ by computing the Chern number of the pull-back of its tangent bundle to a $\mathbb{C}P^1$. Determine the Chern class of a smooth algebraic hypersurface tranversely defined by a homogenous polynomial of degree d.