Homework Set 8

Floer Homology 2019

The following three problems are intended to inspire a discussion. The goal is to understand how to approach these questions. We will thus not present "complete" solutions.

Problem 1
(i) Repeat the definition of Sobolev mappings/functions and their basic properties: embedding into C^k-spaces (when is it compact?), products of Sobolev functions.
(ii) The following fact is useful for the next problems.
Let $U \subset \mathbb{R}^m$ be an open set, $F : U \to \mathbb{R}^n$ be a smooth map. Let $C > 0$, $p > 2$. Show that F induces a map $F : W^{1,p}([-C, C] \times S^1, U) \to W^{1,p}([-C, C] \times S^1, \mathbb{R}^n)$ defined by composition $u\, mapsto F \circ u$. Moreover, this map is smooth.

Problem 2
Let (M, g) be a closed Riemannian manifold. Denote by $\exp_p : T_p \to M$ the exponential map, $p \in M$.

(i) Let $u \in W^{1,p}([-C, C] \times S^1, M)$ and $\xi \in W^{1,p}([-C, C] \times S^1, u^*TM)$. We define $\exp_u(\xi) : [-C, C] \times S^1 \to M$ by

$$\exp_u(\xi)(s, t) := \exp_{u(s,t)}(\xi(s, t)).$$

Show that $\exp_u(\xi)(s, t) \in W^{1,p}([-C, C] \times S^1, M)$ and that for $\epsilon > 0$ small enough the restriction

$$\exp_u : B_\epsilon \to W^{1,p}([-C, C] \times S^1, M)$$

for the ball $B_\epsilon \subset W^{1,p}([-C, C] \times S^1, T^*M)$ is a 1-1 smooth map onto an open neighbourhood U_u of $u \in W^{1,p}([-C, C] \times S^1, M)$.
(ii) The goal is to show that the maps of (i) define a smooth atlas which turn $W^{1,p}([-C, C] \times S^1, M)$ into a smooth Banach manifold. One way could be to show that for $u, v \in ([-C, C] \times S^1, M)$ on the open subset $\exp_u^{-1}(U_u \cap U_v)$ the composition $\exp_v \circ \exp_u^{-1}$ is differentiable.
(iii) Alternatively, one could proceed as follows. Let $M \subset \mathbb{R}^N$ be a smooth embedding, $V \subset \mathbb{R}^N$ a tubular neighborhood of M and $p : V \to M$ be a differentiable map such that $p|_M = \mathrm{id}_M$. Show that

$$\exp_u^{-1} \circ p : p^{-1}(U_u) \subset W^{1,p}([-C, C] \times S^1, \mathbb{R}^N) \to ([-C, C] \times S^1, T^*M)$$

is differentiable.
(iv) Show that the space of maps $\mathcal{B}^p(x, y) := W^{1,p}(\mathbb{R} \times S^1, M)$ consisting of $u : \mathbb{R} \times S^1 \to M$ which are in $u \in W^{1,p}_{\text{loc}}$ and that $\lim_{s \to -\infty} u(s, .) = x : S^1 \to M$ and $\lim_{s \to -\infty} u(s, .) = y : S^1 \to M$ such that for $C > 0$ sufficiently large $\exp_x^{-1}(u|_(- \infty, -C]) \in W^{1,p}((-\infty, -C], x^*TM)$ and $\exp_y^{-1}(u|_[C, \infty)) \in W^{1,p}([C, \infty), y^*TM)$ for given smooth maps x, y is a Banach manifold. x, y are not necessarily periodic solutions of Hamilton's equation.

Problem 3
Let U_u be as in Problem 1. For $v = \exp_u(\xi) \in U_u$, $\xi \in W^{1,p}([-C, C] \times S^1, u^*TM)$ denote by $P_{u,v} : u^*TM \to v^*TM$ the parallel transport along the family of geodesics $\{\tau \mapsto \exp_{u(s,t)}(\tau\xi(s, t))\}_{s,t}$. Then this defines a continuous linear map $P_{u,v} : L^p(u^*TM) \to L^p(v^*TM)$ and via $U_u \times L^p(u^*TM) \overset{\cong}{\to} \mathcal{E}|_{U_u}$ given by $(u, \eta) \to P_{u,v}(\eta)$ a trivialization of the bundle $\mathcal{E}$. Show that the transition maps of this trivialization depend smoothly on the base point.