The Morse-Smale-Witten-Complex

Seminar "Floer Theory"

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1 Introduction

Let M be a compact smooth Riemannian manifold and $f\colon M \to \mathbb{R}$ a Morse function.

Definition. (1) The set of **critical points** of f is denoted $\mathrm{Crit}(f) \coloneqq \{x \in M \mid df(x) = 0\}$.

(2) For $x \in \mathrm{Crit}(f)$, the **Hessian** $d^2 f(x)\colon T_xM \times T_xM \to \mathbb{R}$ is given in local coordinates by the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j}$.

(3) A function $f\colon M \to \mathbb{R}$ is called **Morse function** if every critical point of f is nondegenerate, i.e. $d^2 f(x)$ is nondegenerate (i.e. the above matrix is nonsingular.

Remark. Note that $df^2(x)$ is symmetric, hence diagonalizable. The Hessian is independent of the choice of coordinates if $df(x) = 0$: for the Levi-Civita connection ∇, one has

$$\langle \nabla_\xi \nabla f(x), \eta\rangle = d^2 f(x)(\xi, \eta)$$

where one defines $\nabla^2 f(x)\colon T_xM \to T_xM$ by $\nabla^2 f(x)\xi \coloneqq \nabla_\xi \nabla f(x)$.

Proof. The above equation holds because for $e_i = \frac{\partial}{\partial x_i}$,

$$\begin{aligned}
\langle \nabla_{e_i} \nabla f(x), e_j\rangle &= e_i\left(\langle \nabla f(x), e_j\rangle\right) - \langle \nabla f(x), \nabla_{e_i} e_j\rangle \\
&= e_i\left(df(x)\left(e_j\right)\right) - df(x)\left(\nabla_{e_i} e_j\right) \\
&= \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = df^2(x)\left(e_i, e_j\right)
\end{aligned}$$

□

Consider the negative gradient flow

$$\dot{u} = -\nabla f(u) \tag{1}$$

and denote by $\varphi^t\colon M \to M$ the flow.

Definition/Theorem. The **stable** and **unstable manifolds**

$$
\begin{aligned}
W^s(x,f) &:= \left\{x \in M \mid \lim_{s\to\infty} \varphi^s(z) = x\right\},\\
W^u(x,f) &:= \left\{x \in M \mid \lim_{s\to-\infty} \varphi^s(z) = x\right\}
\end{aligned}
$$

are smooth submanifolds of M for every $x \in \mathrm{Crit}(f)$. The **Morse index** of x is

$$\mathrm{ind}_f(x) := \dim W^u(x,f) = \nu^-\left(df^2(x)\right)$$

where $\nu^-(df^2(x))$ is the number of negative eigenvalues of the matrix of second derivatives of f, counted with multiplicities.

Since $df^2(x)$ is diagonalizable and invertible,

$$\dim W^u(x,f) + \dim W^s(x,f) = \nu^-(df^2(x)) + \nu^+(df^2(x)) = n$$

The gradient flow 1 is called a **Morse-Smale system** if, for any pair $x, y \in \mathrm{Crit}(f)$, the stable and unstable manifolds intersect transversely (i.e. their tangent spaces span the tangent space of M at each point of intersection). In that case the set

$$\mathcal{M}(y,x,f) := W^s(x,f) \cap W^u(y,f)$$

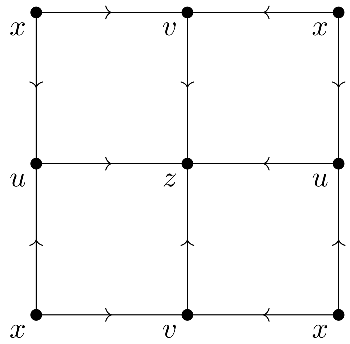
of points in M whose gradient lines connect y to x, is a smooth submanifold of M with dimension

$$
\begin{aligned}
\dim \mathcal{M}(y,x,f) &= n - (n - \dim W^s(x,f)) - (n - \dim W^u(y,f))\\
&= \dim W^s(x,f) + \dim W^u(y,f) - n\\
&= \dim W^s(x,f) - \dim W^s(y,f)\\
&= \mathrm{ind}_f(y) - \mathrm{ind}_f(x).
\end{aligned}
$$

We interpret $\mathcal{M}(y,x,f)$ as the space of gradient flow lines $u\colon \mathbb{R} \to M$ from $y = \lim_{s\to-\infty} u(s)$ to $x = \lim_{s\to\infty} u(s)$. The group $\mathbb{R}$ acts on $\mathcal{M}(y,x,f)$ smoothly, freely and properly, so one can take the quotient $\widehat{\mathcal{M}}(y,x,f)$ which is a manifold of dimension $\mathrm{ind}_f(y) - \mathrm{ind}_f(x) - 1$. Thus $\mathrm{ind}_f(x) < \mathrm{ind}_f(y)$ if there is a connecting orbit from y to x. One can easily see that f is monotonically decreasing along flow lines. In short, both f and the index decrease along flow lines.

An example of a gradient flow that is not Morse-Smale is the torus, embedded in $\mathbb{R}^3$, with Morse function one of the coordinate projections.

Remark. We will see later that the space $\widehat{\mathcal{M}}(y,x,f)$ of gradient flow lines from y to x is a finite set if the index difference is 1. With this in mind, we can define the Morse-Smale-Witten complex.



Definition. The **Morse-Smale-Witten complex** in $\mathbb{Z}_2$ coefficients has the chain groups

$$CM_k(f) = \bigoplus_{\substack{x \in \mathrm{Crit}(f) \\ \mathrm{ind}_f(x)=k}} \mathbb{Z}_2\langle x\rangle$$

with boundary operator $\partial = \partial^M : CM_k(f) \to CM_{k-1}(f)$,

$$\partial^M\langle y\rangle = \sum_{\substack{x \in \mathrm{Crit}(f) \\ \mathrm{ind}_f(x)=k-1}} \#\widehat{\mathcal{M}}(y, x, f)\langle x\rangle$$

Remark. We define the Morse-Smale-Witten complex only for coefficients in $\mathbb{Z}_2$. It can be defined in $\mathbb{Z}$ coefficients but that requires careful consideration of orientations, which we omit here.

Theorem. *Given a Morse-Smale flow with $CM(f)$ and ∂^M defined as above, then ∂^M is well-defined, $\partial^M \circ \partial^M = 0$ and there is a natural isomorphism*

$$HM_k(M, f) = \frac{\ker \partial_k^M}{\mathrm{im}\, \partial_{k+1}^M} \to H_k(M, \mathbb{Z}_2)$$

of the Morse homology and the singular homology of M. In particular the Morse homology does not depend on the choice of the Morse function and metric.

Example. Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the Morse function $f : T^2 \to \mathbb{R}, f(x, y) = \cos(2\pi x) + \cos(2\pi y)$.

We see $\mathrm{Crit}(f) = \{x, u, v, z\}$ for $x = (0.5, 0.5), u = (0.5, 0), v = (0, 0.5), z = (0, 0)$. The chain groups are given by

$$CM_2(f) = \mathbb{Z}_2\langle x\rangle,$$
$$CM_1(f) = \mathbb{Z}_2\langle u\rangle + \mathbb{Z}_2\langle v\rangle,$$
$$CM_0(f) = \mathbb{Z}_2\langle z\rangle$$

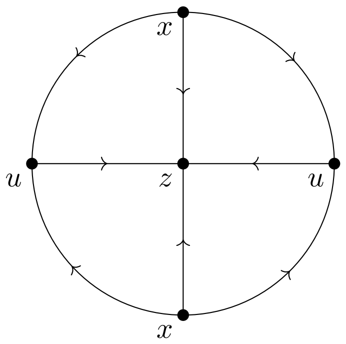


Figure 1.1: A Morse-Smale gradient flow on the real projective plane.

And the boundary operators are given by:

$$\partial\langle x\rangle = 2\langle u\rangle + 2\langle v\rangle = 0$$
$$\partial\langle u\rangle = 2\langle z\rangle = 0, \partial\langle v\rangle = 2\langle z\rangle = 0$$
$$\partial\langle z\rangle = 0.$$

Hence we obtain $H_2(T^2, \mathbb{Z}_2) = \mathbb{Z}_2, H_2(T^2, \mathbb{Z}_2) = \mathbb{Z}_2^2, H_2(T^2, \mathbb{Z}_2) = \mathbb{Z}_2$.

Example. Consider the $\mathbb{R}P^2$ with gradient flow as in 1.1. We have critical points x, u, z.

$$CM_2(f) = \mathbb{Z}_2\langle x\rangle,$$
$$CM_1(f) = \mathbb{Z}_2\langle u\rangle,$$
$$CM_0(f) = \mathbb{Z}_2\langle z\rangle$$

And the boundary operators are given by:

$$\partial\langle x\rangle = 2\langle u\rangle = 0$$
$$\partial\langle u\rangle = 2\langle z\rangle = 0$$
$$\partial\langle z\rangle = 0.$$

Hence $H_i(\mathbb{R}P^2, \mathbb{Z}_2) = \mathbb{Z}_2$ for $i = 0, 1, 2$.

Sketch of proof. We sketch the proof of $\partial^M \circ \partial^M = 0$ and that $\widehat{\mathcal{M}}(y, x)$ is finite if $\mathrm{ind}_f(y) = \mathrm{ind}_f(x) - 1$. Showing that $\partial^2 = 0$ is equivalent to showing that for each $x, z \in \mathrm{Crit}(f)$ with $\mathrm{ind}_f(z) - \mathrm{ind}_f(x) = 2$,

$$\sum_{\substack{y\in \mathrm{Crit}(f) \\ \mathrm{ind}_f(y)=k}} \#\widehat{\mathcal{M}}(y, x, f)\#\widehat{\mathcal{M}}(z, y, f) = \# \bigcup_{\substack{y\in \mathrm{Crit}(f) \\ \mathrm{ind}_f(y)=k}} \widehat{\mathcal{M}}(y, x, f) \times \widehat{\mathcal{M}}(z, y, f) \in 2\mathbb{Z}.$$

Consider the one dimensional manifold $\widehat{\mathcal{M}}(z, x, f)$. It suffices to show that the ends of this space are in a bijection with the above set. Since the boundary points of a compact 1-manifold come in pairs, this would prove the proposition.

We define a compactification of $\widehat{\mathcal{M}}(y,x,f)$ by

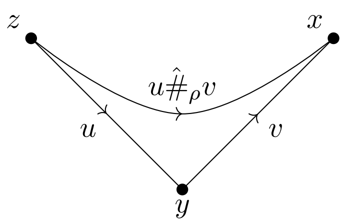
$$\widehat{\mathcal{M}}^C(y,x,f) \coloneqq \bigcup_{x_i \in \mathrm{Crit}(f)} \widehat{\mathcal{M}}(y,x_1,f) \times \widehat{\mathcal{M}}(x_1,x_2,f) \times \ldots \times \widehat{\mathcal{M}}(x_k,x,f)$$

It is possible (although not easy) to equip this with a topology such that it is compact and $\widehat{\mathcal{M}}(y,x,f) \subseteq \widehat{\mathcal{M}}^C(y,x,f)$ is a dense open subspace.

From this it follows in particular that for $\mathrm{ind}_f(y) - \mathrm{ind}_f(x) = 1$, the space $\widehat{\mathcal{M}}(y,x,f) \subseteq \widehat{\mathcal{M}}^C(y,x,f)$ is also compact and zero-dimensional, hence finite, so ∂^M is well-defined. For $\mathrm{ind}_f(y) - \mathrm{ind}_f(x) = 2$ we see that $\widehat{\mathcal{M}}^C(y,x,f)$ is a compact 1-dimensional manifold with boundary. In the boundary we can only have simply broken trajectories: there cannot be more than one intermediate critical point because the index decreases strictly monotonically along flow lines, and if the trajectory were not broken then it would be part of the interior.

The density part of the above proposition is called the gluing theorem. We state this explicitly because a similar statement will appear later in the seminar. The following is a special case of theorem 3 as given in Schwarz, M.: Morse Homology.

Theorem. *Given $(u,v) \in \widehat{\mathcal{M}}(z,y) \times \widehat{\mathcal{M}}(y,x)$, there is a smooth map $\hat{\#}\colon [\rho_{(u,v)}, \infty) \to \widehat{\mathcal{M}}(z,x)$ mapping ρ to $u \hat{\#}_\rho v$ such that $u \hat{\#}_\rho v \to (u,v)$ in $\widehat{\mathcal{M}}^C(z,x)$ as $\rho \to \infty$. On the other hand, any sequence of unparametrized trajectories converging to a simply broken trajectory eventually lies in the range of such a gluing map $\hat{\#}$.*



Thus we see that the

$$\partial\left(\widehat{\mathcal{M}}^C(z,x)\right) = \# \bigcup_{\substack{y \in \mathrm{Crit}(f) \\ \mathrm{ind}_f(y) = k}} \widehat{\mathcal{M}}(y,x,f) \times \widehat{\mathcal{M}}(z,y,f)$$

This concludes the proof that $\partial^2 = 0$. $\square$

What follows is an intuitive argument why the homologies should be isomorphic. Consider again the torus example from above. One can equip it with a cw-complex structure such that every cell interior of a k-cell is given by $W^u(y,f)$ for some $y \in \mathrm{Crit}(f)$, $\mathrm{ind}(y) = k$. One checks by hand that the assignment $\langle y \rangle \mapsto \langle \text{cell of } W^u(y,f) \rangle$ gives an isomorphism of $CM_k(f) \to C^{cw}(T^2)$ and that this is a chain map, i.e. it preserves the boundary operator, hence we have an isomorphism of $HM_k(T^2,f) \to H_k^{cw}(T^2)$.

This however does not easily give a general proof of the isomorphism. One could hope to obtain in general a cell decomposition on M with the interiors of cells given by unstable manifolds of critical points and to show that the resulting chain complex is isomorphic to the Morse Smale Witten complex. The issue is with constructing the cell decomposition. One can show using φ^t that the unstable manifolds are homeomorphic to open balls. However to construct a cw complex we need a map from a closed ball to a subset of M, which on the interior is a homeomorphism. Notice that $\lim_{t\to\infty} \varphi^t(x) \in \operatorname{Crit}(f)$, so φ^t cannot map to the entire boundary as $t \to \infty$. Hence we cannot directly use the flow map to construct the cw structure on M.

Corollary (Morse-Inequalities)**.** *Let $f\colon M \to \mathbb{R}$ be a Morse function and denote $c_k = \#\{x \in \operatorname{Crit}(f) \mid \operatorname{ind}_f(x) = k\}$ and $b_k = \operatorname{rank} H_k(M, \mathbb{Z})$ the k-th Betti number. Then*

$$c_k - c_{k-1} + \cdots + (-1)^k c_0 \geq b_k - b_{k-1} + \cdots + (-1)^k b_0$$

for $0 \leq k \leq n = \dim M$ with equality for $k = n$ and $c_k \geq b_k$.

Proof. Because $b_k = \operatorname{rank}\ker\partial_k - \operatorname{rank}\operatorname{im}\partial_{k+1}$ and $c_k = \operatorname{rank}\ker\partial_k + \operatorname{rank}\operatorname{im}\partial_k$,

$$\begin{aligned} &c_k - c_{k-1} + \cdots + \pm c_0 = \operatorname{rank} CM_k(f) - \operatorname{rank} CM_{k-1}(f) + \cdots \pm \operatorname{rank} CM_0(f) \\ &= \operatorname{rank}\ker\partial_k + \operatorname{rank}\operatorname{im}\partial_k - \operatorname{rank}\ker\partial_{k-1} - \operatorname{rank}\operatorname{im}\partial_{k-1} + \cdots + \\ &\quad \mp \operatorname{rank}\ker\partial_1 \mp \operatorname{rank}\operatorname{im}\partial_1 \pm \operatorname{rank}\ker\partial_0 \pm \operatorname{rank}\operatorname{im}\partial_0 \\ &= \operatorname{rank}\operatorname{im}\partial_{k+1} + b_k - b_{k-1} + \cdots \pm b_0 \end{aligned}$$

□