Conley-Zehnder index of critical points and computing the Fredholm index

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Abstract

Given a time-dependent Hamiltonian system whose periodic orbits are non-degenerate, we define the Conley-Zehnder index of a critical points of its symplectic action functional. As shown previously, the linearisation of the Floer map is a Fredholm operator; we compute its Fredholm index (which is a difference of Conley-Zehnder indices).

1 Context and motivation

The motivation for this talk is twofold.

The first line of support comes from analogy to Morse theory, as much of this course is devoted to generalising the methods of Morse theory to the infinite-dimensional setting. We recall that Morse homology of Morse-Smale system was defined via the Morse-Smale-Witten complex, consisting of free modules over the non-degenerate critical points. In particular, we needed a grading of critical points which was provided via their Morse index. We will also need such a grading for defining Floer homology. Hence, we would like to assign an index to each critical point of our action functional.

The second line of motivation comes from our context within the program of defining Floer homology: given a time-dependent Hamiltonian system, we considered its non-degenerate 1-periodic orbits, which are exactly the critical points of symplectic action functional. We defined a metric on its domain (which was the space of contractible loops) and then considered the negative gradient flow given by the action functional. Its defining equation is the Floer equation. We were investigating whether its solution set is a finite-dimensional manifold.

This will be completed in the next talk (the previous talk proved major parts): for generic pairs (H, J), we will have proved that the moduli space

$M(x, y; H, J)$ of finite-energy solutions connecting the periodic orbits x and y is a finite-dimensional manifold.

The basic approach was to regard the moduli space as the zero set of the Floer map $\mathcal{F}$ between infinite-dimensional Banach spaces and use the implicit function theorem. Checking its hypotheses, we need to show that the linearisation $d\mathcal{F}$ of $\mathcal{F}$ is surjective and has a bounded right inverse. We just finished the proof that $d\mathcal{F}$ is a Fredholm operator. The last step will be to apply the Sard-Smale theorem, which basically proves that the hypotheses are satisfied for generic (H, J).

With today's talk, first of all, we follow our Pavlov reflex ("given a Fredholm operator, can we determine its index?"). Secondly, we are computing the dimension of the moduli space: by the implicit function theorem, we have $\dim M(x, y; H, J) = \dim \ker d\mathcal{F}$. Since the linearisation is surjective, this equals the Fredholm index of the linearisation.

2 Outline

Accordingly, this talk has two parts. We begin by assigning an index to each non-degenerate critical point of our action functional, the *Conley-Zehnder index*. Then, we compute the Fredholm index of $d\mathcal{F}$ which equals the dimension of the moduli space $\mathcal{M}(x, y; H, J)$. In analogy to Morse theory, we expect this to be the difference of the orbits' indices $\mu(y) - \mu(x)$.

This will be true if the underlying closed symplectic manifold (M, ω) is aspherical, i.e., that $\int_{\mathbb{S}^2} u^*\omega = 0$ for all smooth maps $u \colon \mathbb{S}^2 \to M$. If this is not the case, the dimension of the moduli space near a solution u also depends on the energy of u!

For defining the Conley-Zehnder index, we assign to each non-degenerate 1-periodic orbit x of our Hamiltonian system a path in the *symplectic group*, the group formed by all *symplectic matrices*, and determine the index from that. (Recall that the Hamiltonian flow preserves the symplectic form, in coordinates, such maps are exactly given by symplectic matrices.) Since we will use quite a few properties of this group, we will start by discussing these.

All material in this document is classical. All material in this document is taken from [Sal99] and [AD14], with the background about the symplectic group being only found in [AD14]. We also found it helpful to consult [MS17] regarding the properties of the symplectic group.

3 Symplectic matrices and their properties

3.1 Basic definitions and subgroups

We are given our symplectic manifold (M, ω) and want to study maps preserving this symplectic form in coordinates, i.e. locally. By Darboux' theorem, every symplectic manifold locally looks with $\mathbb{R}^{2n}$ with standard symplectic form, hence it suffices to consider that.

Let (R^{2n}, ω) be the standard symplectic space. Denote the standard complex structure by J_0; this corresponds to the $2n \times 2n$-matrix $J_0 = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$.

Definition 3.1. *A linear map $g\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called* symplectic *iff it satisfies*

$$\omega(gZ, gZ') = \omega(Z, Z') \qquad \text{for all } Z, Z' \in \mathbb{R}^{2n}.$$

Observe that g is symplectic iff its matrix A in the standard basis satisfies $A^t J_0 A = J_0$.

Definition 3.2. *A real $2n \times 2n$-matrix $A \in M(2n; \mathbb{R})$ is called* symplectic *iff it satisfies $A^t J_0 A = J_0$.*

Proposition 3.3. *The set $\mathrm{Sp}(2n)$ of all symplectic $2n \times 2n$-matrices forms a subgroup of $M(2n; \mathbb{R})$ called the* symplectic group. ☐

The symplectic group has a few noteworthy relations with other matrix Lie groups. We consider the groups $\mathrm{O}(2n)$ of orthogonal real $2n \times 2n$-matrices, $\mathrm{GL}(n, C)$ of invertible complex $n \times n$ matrices and $\mathrm{U}(n)$ of unitary matrices as subgroups of $\mathrm{GL}(2n, \mathbb{R})$.

Proposition 3.4. *We have the equalities*

$$\mathrm{Sp}(2n) \cap O(2n) = \mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = O(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{U}(n).$$

Proof. is skipped since just routine computation. ☐

For later reference, we note that the intersection $\mathrm{Sp}(2n) \cap \mathrm{O}(2n)$ consists of the matrices

$$\begin{pmatrix} U & -V \\ V & U \end{pmatrix} \in \mathrm{GL}(2n; \mathbb{R}),$$

with

$$U^t V = V^t U \text{ and } U^t U + V^t V = \mathrm{Id}\,.$$

This is exactly the condition for $U + iV$ to be a unitary matrix.

Proposition 3.5. *For all $A \in \mathrm{Sp}(2n)$, we have* $\det A = 1$.

Proof. The relation $A^t J_0 A = J_0$ immediately implies $\det A = \pm 1$ holds. The fact that $\det A = 1$ will follow e.g. from the fact that the symplectic group is path-connected, which we will show later, as $\det(Id) = 1$.

From a higher-level point of view, $\det(A) > 0$ just means that A is orientation preserving, which follows since the standard symplectic form is compatible with the almost complex structure J_0. □

3.2 Eigenspaces of symplectic matrices

In the next subsection, we will also need some results about eigenspaces of symplectic matrices. In order to work over an algebraically closed field, so that every characteristic polynomial splits into linear factors, we complexify: we consider symplectic matrices as elements of $\mathrm{GL}(n, \mathbb{C})$ and extend the symplectic form ω to a bilinear map $\omega\colon \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$ using J_0.

Proposition 3.6. *If λ, μ are distinct eigenvalues of $A \in \mathrm{Sp}(2n)$ with $\lambda\mu \neq 1$ and $r, s \geq 1$, the spaces $\ker(A - \lambda\,\mathrm{Id})^r$ and $\ker(A - \mu\,\mathrm{Id})^s$ are orthogonal w.r.t. ω.*

We skip the proof, since nothing interesting happens there: it just goes by induction over r and s, using that A is symplectic and the definition of eigenvalues. □

We define generalised eigenspaces E_λ for eigenvalues λ by

$$E_\lambda = \bigcup_{m \in \mathbb{N}} \ker(A - \lambda\,\mathrm{id})^m \subset \mathbb{C}^{2n}.$$

Recall from linear algebra that each generalised eigenspace E_λ is a vector space whose dimension equals the algebraic multiplicity of the eigenvalue λ, and that $\mathbb{C}^{2n}$ is the direct sum of these eigenspaces.

Corollary 3.7. *If $\lambda\mu \neq 1$, we have $\omega(E_\lambda, E_\mu) = 0$.*
The restrictions of ω to E_1 and E_{-1} are non-degenerate. For every eigenvalue $\lambda \neq \pm 1$, the restriction of ω to $E_\lambda \bigoplus E_{\lambda^{-1}}$ is non-degenerate.

Proof. The first phrase is a clear corollary. The second is clear by definition, since A is symplectic; the third is analogous. □

3.3 Eigenvalues of symplectic matrices

In a precise sense, eigenvalues of symplectic matrices come in pairs.

Proposition 3.8. *If $A \in \mathrm{Sp}(2n)$, a complex number $\lambda \in \mathbb{C}$ is an eigenvalue of A iff λ^{-1} is.*

Proof. We have $A^t J_0 A = J_0 \Leftrightarrow A^t = J_0 A^{-1} J_0$, hence A^t and A^{-1} are conjugate and in particular have the same eigenvalues. □

In fact, also the algebraic multiplicities of the eigenvalues coincide.

Proposition 3.9. *For $A \in \mathrm{Sp}(2n)$, we have* $\det(A - \lambda \,\mathrm{Id}) = \lambda^{2n} \det(A - \lambda^{-1})$.

Proof. Just a slightly clever computation, using $\det A = 1$: we have

$$\begin{aligned}
\det(A - \lambda \,\mathrm{id}) &= \det(-J_0(A^{-1})^t J_0 - \lambda \,\mathrm{id}) \\
&= \det(J_0(-(A^{-1})^t)J_0 + \lambda J_0^2) \\
&= \det(-(A^{-1})^t + \lambda \,\mathrm{id}) \\
&= \det((A^{-1})^t) \det(\mathrm{id} - \lambda A^t) \\
&= \det(\lambda A^t - Id) = \lambda^{2n} \det(A - \lambda^{-1} \,\mathrm{id}).
\end{aligned}$$

□

Corollary 3.10. *For all $A \in \mathrm{Sp}(2n)$ and $\lambda \in \mathbb{C}$, the algebraic multiplies of λ and λ^{-1} coincide, and equal the algebraic multiplicity of $\overline{\lambda}$ and $\overline{\lambda^{-1}}$.*

3.4 Topology of the symplectic group

The purpose of this subsection is to sketch the proof of the following result.

Proposition 3.11. *The group* $\mathrm{Sp}(2n)$ *is path-connected and* $\pi_1(\mathrm{Sp}(2n)) \cong \mathbb{Z}$.

Our strategy is to show that $\mathrm{Sp}(2n)$ deformation retracts to the unitary group $\mathrm{U}(n)$: it is a standard fact that $\mathrm{U}(n)$ is path-connected and has fundamental group $\mathbb{Z}$, and a deformation retract is in particular a homotopy equivalence.

To put this into context: the unitary group $\mathrm{U}(n)$, seen as a submanifold of $\mathrm{GL}(2n, \mathbb{R})$, has dimension n^2, whereas the symplectic group $\mathrm{Sp}(2n)$ is a submanifold of dimension $n(2n + 1) = 2n^2 + n$.

The first ingredient is the so-called *polar decomposition*. Recall that any complex square matrix A has a polar decomposition, i.e. can be written as a product $A = UP$ of a unitary matrix U and a positive semi-definite Hermitian matrix P. Further recall that if A is invertible, U and P are uniquely determined: $P = \sqrt{A^*A}$ is the principal square root of the positive semi-definite matrix A^*A, and U is accordingly given as $U = AP^{-1}$.
Note that this is just for analogy: we will use a slightly different form!

Sketch of proof. The polar decomposition in $\mathrm{GL}(2n, \mathbb{R})$, in a different version than above, yields a homeomorphism

$$\begin{aligned}
\mathrm{Sp}(2n) &\longrightarrow \mathrm{U}(n) \cdot C_n \\
A &\longmapsto U \cdot S \qquad \text{with } S = \sqrt{A^t A},\ U = AS^{-1},
\end{aligned}$$

where $C_n = \{\text{positive definite symmetric symplectic matrices}\} \subset \mathrm{Sp}(2n)$ is an open subset of the symmetric symplectic matrices. The idea is to show that C_n is contractible, yield will yield a deformation retract.

Any matrix $A \in C_n$ has positive real eigenvalues. In particular, it has a unique real logarithm $\log A$ and the mapping $A \mapsto \log A$ is continuous. This logarithm can e.g. be constructed by diagonalising A and taking the logarithm of entries component-wise. This construction shows that $\log A$ is also symmetric and positive definite.

Now, the map $\phi\colon C_n \times [0,1] \ni (A,t) \mapsto \exp(t \log A) \to \mathrm{Sp}(2n)$ yields a contraction of C_n: it only remains to show that $\operatorname{im}\phi \subset \mathrm{Sp}(2n)$. Using Corollary 3.7, one can show that A has a symplectic basis in which it is diagonal. This implies that $\exp(t \log A)$ is also symplectic. □

References

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