

# BSDEs and their connection to stochastic control, Convex BSDEs

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## Theorem (Linear BSDEs)

Consider the LBSDE

$$-dY_t = [\phi_t + Y_t\beta_t + Z_t\gamma_t]dt - Z_t dW_t, \quad Y_T = \xi$$

Then  $Y_t$  is given by

$$Y_t = E\left[\xi\Gamma_T^t + \int_t^T \Gamma_s^t \phi_s ds \mid \mathcal{F}_t\right] \quad \mathbb{P} \text{ a.s.}$$

where  $\Gamma_s^t$  is the **adjoint process** defined for  $s \geq t$  by the (forward) LSDE

$$d\Gamma_s^t = \Gamma_s^t[\beta_s ds + \gamma_s dW_s], \quad \Gamma_t^t = 1$$

## Theorem (Comparison Theorem)

Let  $(f^1, \xi^1)$  and  $(f^2, \xi^2)$  be standard parameters of BSDEs with corresponding square-integrable solutions  $(Y^1, Z^1)$  and  $(Y^2, Z^2)$ . Suppose that

$$\xi^1 \geq \xi^2 \quad \mathbb{P} \text{ a.s.}$$

$$f^1(t, Y_t^2, Z_t^2) - f^2(t, Y_t^2, Z_t^2) \geq 0 \quad d\mathbb{P} \otimes dt \text{ a.s.}$$

Then we have almost surely

$$Y_t^1 \geq Y_t^2 \quad \forall t \in [0, T]$$

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Consider the following **stochastic control** problem.

Let  $\mathcal{U} := \{u : \Omega \times [0, T] \rightarrow \mathbb{R} : u \text{ is predictable}\}$  be the set of feasible **controls**.

Let  $\{H^u\}_{u \in \mathcal{U}}$  be a family of processes, defined via the the SDEs

$$dH_t^u = H_t^u[d(t, u_t)dt + n(t, u_t)dW_t], \quad H_0^u = 1$$

The problem is to minimize, over all feasible controls, the **objective function**

$$J(u) := E\left[\int_0^T H_t^u k(t, u_t)dt + H_T^u K(u_T)\right]$$

that is, find the **value function**

$$Y_0^* := \inf_{u \in \mathcal{U}} J(u)$$

With

$$f^u(t, y, z) = k(t, u_t) + d(t, u_t)y + n(t, u_t)^*z \quad \xi^u = K(u_T)$$

we can see that for the process  $(Y^u, Z^u)$  that solves the LBSDE  $(f^u, \xi^u)$ :

$$J(u) = Y_0^u$$

$H^u$  corresponds to the adjoint process associated with  $(Y^u, Z^u)$ .

## Theorem (Verification Theorem)

The parameters  $(f, \xi)$  defined by

$$f(t, y, z) := \operatorname{ess\,inf}\{f^u(t, y, z) : u \in \mathcal{U}\}, \quad \xi := \operatorname{ess\,inf}\{\xi^u : u \in \mathcal{U}\}$$

are standard parameters. Let  $(Y, Z)$  be the solution of the BSDE associated with  $(f, \xi)$ . Then:

*$Y$  is the value function  $Y^*$  of the control problem; that is for each  $t \in [0, T]$*

$$Y_t = Y_t^* = \operatorname{ess\,inf}\{Y_t^u : u \in \mathcal{U}\}$$

We are first going to show a more general result.

## Lemma

Let  $(f, \xi)$  and  $(f^\alpha, \xi^\alpha)$  be a family of standard parameters and let  $(X, Y)$  and  $(X^\alpha, Y^\alpha)$  be the solution of associated BSDEs.

Suppose that

- ▶  $f^\alpha$  are equi-Lipschitz with constant  $C$ , i.e.  $d\mathbb{P} \otimes dt$  a.s.:

$$\forall \alpha : |f^\alpha(w, t, y_1, z_1) - f^\alpha(w, t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|)$$

- ▶ for each  $\epsilon > 0$ , there exists an index  $\alpha^\epsilon$  s.t.

$$f(t, Y_t, Z_t) = \text{ess inf } f^\alpha(t, Y_t, Z_t) \geq f^{\alpha^\epsilon}(t, Y_t, Z_t) - \epsilon, d\mathbb{P} \otimes dt \text{ a.s.} \quad (1)$$

$$\xi = \text{ess inf } \xi^\alpha \geq \xi^{\alpha^\epsilon} - \epsilon, \mathbb{P} \text{ a.s.}$$

Then

$$Y_t = \text{ess inf } Y_t^\alpha \quad \forall t \in [0, T], \quad \mathbb{P} \text{ a.s.}$$



## Theorem (Verification Theorem)

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$$f(t, y, z) := \operatorname{ess\,inf}\{f^u(t, y, z) : u \in \mathcal{U}\}, \quad \xi := \operatorname{ess\,inf}\{\xi^u : u \in \mathcal{U}\}$$

are standard parameters. Let  $(Y, Z)$  be the solution of the BSDE associated with  $(f, \xi)$ . Then:

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$$Y_t = Y_t^* = \operatorname{ess\,inf}\{Y_t^u : u \in \mathcal{U}\}$$

## Proof of Verification Theorem

Let  $\varepsilon > 0$  be given. For  $(w, t) \in \Omega \times [0, T]$  fixed, the sets

$$\begin{aligned} &\{u \in \mathbb{R} : f(t, Y_t(w), Z_t(w)) \geq k(t, w, u) + d(t, w, u)Y_t(w) \\ &\quad + n(t, w, u)Z_t(w) - \varepsilon\} \\ &\{u \in \mathbb{R} : \xi(w) \geq K(w, u) - \varepsilon\} \end{aligned}$$

are nonempty. Since  $Y$  and  $Z$  are predictable, a measurable selection theorem yields that there exists a  $\mathbb{R}$ -valued predictable process  $u^\varepsilon$  s.t.

$$\begin{aligned} f(t, Y_t, Z_t) &= \text{ess inf } f^u(t, Y_t, Z_t) \geq f^{u^\varepsilon}(t, Y_t, Z_t) - \varepsilon \, d\mathbb{P} \otimes dt \text{ a.s.} \\ \xi &= \text{ess inf } \xi^u \geq \xi^{u^\varepsilon} - \varepsilon \end{aligned}$$

One immediate consequence of this theorem is

### Corollary (Optimality Criterion)

*A control  $u_s^0$  is optimal if and only if*

$$\begin{aligned} f(s, Y_s, Z_s) &= f^{u^0}(s, Y_s, Z_s) \quad d\mathbb{P} \otimes ds \text{ a.s.} \\ \xi &= \xi^{u^0} \quad \mathbb{P} \text{ a.s.} \end{aligned}$$

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Let  $f(t, y, z)$  be a standard generator of a BSDE convex w.r.t.  $(y, z)$  and let  $F(t, \beta, \gamma)$  be the **polar process** associated with  $f$ :

$$F(w, t, \beta, \gamma) := \inf_{(y, z) \in \mathbb{R} \times \mathbb{R}} [f(w, t, y, z) - \beta y - \gamma z]$$

Its **effective domain** is

$$\mathcal{D}_F^t := \{(w, t, \beta, \gamma) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} : F(w, t, \beta, \gamma) > -\infty\}$$

And we have

$$f(w, t, y, z) = \sup\{F(w, t, \beta, \gamma) + \beta y + \gamma z : (\beta, \gamma) \in \mathcal{D}_F^t\}$$

Now let

$$f^{\beta, \gamma}(t, y, z) := F(t, \beta_t, \gamma_t) + \beta_t y + \gamma_t z$$

where  $(\beta, \gamma)$  are predictable processes.

For these to be standard generators, we can only allow processes belonging to

$$\mathcal{A} := \{(\beta, \gamma) \in \mathcal{P} : E \int_0^T F(t, \beta_t, \gamma_t)^2 dt < +\infty\}$$

## Lemma

There exists an  $(\bar{\beta}, \bar{\gamma}) \in \mathcal{A}$  s.t.

$$f(t, Y_t, Z_t) = f^{\bar{\beta}, \bar{\gamma}}(t, Y_t, Z_t), \quad d\mathbb{P} \otimes dt \text{ a.s.}$$

## Proof of the Lemma

- ▶  $f$  is uniformly Lipschitz and concave, hence the supremum in the conjugacy relation is actually achieved (for fixed  $(w, t)$ ).
- ▶  $f(\cdot, Y, Z)$ ,  $Y$  and  $Z$  are predictable processes, hence by a measurable selection theorem it follows, that there exists a pair of predictable (bounded) processes  $(\bar{\beta}, \bar{\gamma})$  s.t.

$$f(t, Y_t, Z_t) = f^{\bar{\beta}, \bar{\gamma}}(t, Y_t, Z_t), \quad d\mathbb{P} \otimes dt \text{ a.s.}$$

## Theorem

Let  $f$  be a convex standard generator and  $f^{\beta,\gamma}$  be the associated linear standard generators, with

$$f = \text{ess sup}\{f^{\beta,\gamma} : (\beta, \gamma) \in \mathcal{A}\}, \quad d\mathbb{P} \otimes dt \text{ a.s.}$$

Then  $\mathbb{P}$ -a.s.

$$Y_t = \text{ess sup}\{Y_t^{\beta,\gamma} : (\beta, \gamma) \in \mathcal{A}\}$$

## Proof of Theorem

- ▶ Because of the Comparison Theorem we have for every  $(\beta, \gamma)$ :  
 $Y_t \geq Y_t^{\beta, \gamma}$ . Hence  $Y_t \geq \text{ess sup}\{Y_t^{\beta, \gamma}\}$ .
- ▶ From the uniqueness theorem of BSDEs we have that  
 $Y_t = Y_t^{\bar{\beta}, \bar{\gamma}}$ . Hence  
 $\text{ess sup}\{Y_t^{\beta, \gamma}\} \geq Y_t^{\bar{\beta}, \bar{\gamma}} = Y_t \geq \text{ess sup}\{Y_t^{\beta, \gamma}\}$ .



Thank you!