Pathwise Itô Calculus and introduction to Stochastic Finance in Continuous Time

Lecture Notes

Dirk Becherer[1]

Institut für Mathematik
Humboldt Universität zu Berlin

this version January 30, 2017

[1]Feedback welcome. If you spot a typo, please email me.

Contents

1. Pathwise Itô Calculus

Definition 1.1. *A partition (of time) of* $[0,+\infty)$ *is a set* $\Pi=\{t_0,t_1,\ldots\}$ *with* $0=t_0<t_1<\cdots$ *and* $\lim\limits_{n\to+\infty} t_n=+\infty$*. The mesh-size of* Π *is*

$$|\Pi| = \sup_i |t_{i+1}-t_i| \,.$$

Definition 1.3. *Let* $X\in\mathcal{C}\left([0,+\infty),\mathbb{R}^d\right)$*, with* $X=\left(X^j\right)_{j=1,\ldots,d}$*.* X *is said to be of continuous quadratic variation if*

$$\langle X^j,X^k\rangle_t := \lim_{n\to\infty}\sum_{\substack{t_i\in\Pi_n\\ t_i\le t}}\left(X^j_{t_{i+1}\wedge t}-X^j_{t_i}\right)\left(X^k_{t_{i+1}\wedge t}-X^k_{t_i}\right) \tag{1.1}$$

exists for all $j,k\in\{1,\ldots,d\}$*,* $\forall t\ge 0$*, is continuous in* t*, and is of finite variation.*
In that case, $\langle X^j,X^k\rangle_t$ *is called the (quadratic) covariation of* X^j *and* X^k *until* t*.*

Lemma 1.9. *For* $g\in\mathcal{C}\left([0,+\infty),\mathbb{R}\right)$*, we have*

$$\lim_{n\to\infty}\sum_{\substack{t_i\in\Pi_n\\ t_i\le t}} g(t_i)\Delta_i X^j\Delta_i X^k = \int_0^t g(s)d\langle X^j,X^k\rangle_s$$

Lemma 1.10. *If* X^j *is continuous and of finite variation* $\forall j$*, then*

$$\langle X^j\rangle_t = \langle X^j,X^j\rangle_t = 0 \quad \forall t,$$

in particular $X=(X^j)$ *is then of continuous quadratic variation.*

Theorem 1.11 (d-dimensional Itô Formula)**.** *Let* $X=\left(X^i\right)_{i=1,\ldots,d}$ *be continuous, of continuous quadratic variation, and* $f\in\mathcal{C}^2\left(\mathbb{R}^d,\mathbb{R}\right)$*, then*

$$f(X_t) = f(X_0) + \sum_{j=1}^d \int_0^t \frac{\partial f}{\partial x^j}(X_s)dX^j_s + \frac{1}{2}\sum_{j,k=1}^d\int_0^t \frac{\partial^2 f}{\partial x^j x^k}(X_s)d\langle X^j,X^k\rangle_s, \tag{1.2}$$

with

$$\int_0^t \triangledown f(X_s)dX_s := \sum_{j=1}^d\int_0^t\frac{\partial f}{\partial x^j}(X_s)dX^j_s := \lim_{n\to\infty}\sum_{j=1}^d\sum_{\substack{t_i\in\Pi_n\\ t_i\le t}}\frac{\partial f}{\partial x^j}(X_{t_i})\Delta_i X^j$$

defining the Itô integral as limit of "non-anticipating Riemann sums".

For convenience, (1.2) is often written in differential notation as

$$df(X_t) = \sum_{j=1}^d\frac{\partial f}{\partial x^j}(X_t)dX^j_t + \frac{1}{2}\sum_{j,k=1}^d\frac{\partial^2 f}{\partial x^j x^k}(X_t)d\langle X^j,X^k\rangle_t. \tag{1.3}$$

Corollary 1.13 (Product rule)**.** *For* $\begin{pmatrix} X\\ Y\end{pmatrix}$ *continuous, of continuous quadratic variation (in* $\mathbb{R}^2$*), we have*

$$X_tY_t - X_0Y_0 = \left(\int_0^t X_s dY_s + \int_0^t Y_s dX_s\right) + \langle X,Y\rangle_t \tag{1.4}$$

Lemma 1.14. *Given X and Y two processes which are continuous and of continuous quadratic variation, with $\langle X,Y\rangle_t = \lim_{n\to\infty} \sum_{\substack{t_i\in\Pi_n \\ t_i\le t}} \Delta_i X \Delta_i Y$, then*

a) The quadratic covariation $\langle X,Y\rangle$ exists and is continuous if and only if the quadratic variation $\langle X+Y\rangle$ of $X+Y$ exists and is continuous.

b) (Bilinearity)

- $\langle X,Y\rangle = \langle Y,X\rangle$,
- $\langle aX,Y\rangle = a\langle X,Y\rangle$ *for* $a\in\mathbb{R}$,
- $\langle X^1+X^2,Y\rangle = \langle X^1,Y\rangle + \langle X^2,Y\rangle$ *for* X^1,X^2,X^1+X^2 *of continuous quadratic variation.*

Note that the bilinear form $\langle,\rangle$ is not a scalar product. In fact, a non-zero finite variation continuous function has zero quadratic variation, and so $\langle,\rangle$ is not positive-definite.

c) (Polarization)

$$\langle X,Y\rangle = \frac{1}{2}\left(\langle X+Y\rangle - \langle X\rangle - \langle Y\rangle\right) \tag{1.5}$$

d) (Cauchy-Schwartz)

$$|\langle X,Y\rangle| \le \sqrt{\langle X\rangle\langle Y\rangle} \tag{1.6}$$

Corollary 1.15. *If $\langle X,Y\rangle$ is continuous, then it is of finite variation*

Lemma 1.17. *Let X be $\mathbb{R}$-valued, continuous and of continuous quadratic variation.*
Let $g\in\mathcal{C}^1(\mathbb{R},\mathbb{R})$, then the map $t\mapsto g(X_t)$ is of continuous quadratic variation and

$$\langle g(X_t)\rangle_t = \int_0^t \left(g'(X_s)\right)^2 d\langle X\rangle_s \tag{1.7}$$

In particular, the class of real valued continuous X of continuous quadratic variation is stable under continuously differentiable functions.

Lemma 1.18. *If A is continuous and of finite variation, then $\langle X+A\rangle = \langle X\rangle$*

Theorem 1.19 (Properties of the Itô integral)**.** *Let X be real valued, continuous and of continuous quadratic variation, then*

a) For $g\in\mathcal{C}^1$, the Itô integral $Y_t = \displaystyle\int_0^t g(X_s)dX_s$ is well-defined.

b) The map $t\mapsto Y_t$ is continuous.

c) the map $g\mapsto \int_0^\cdot g(X_s)dX_s$ is linear

d) Y has quadratic variation $\langle Y\rangle_t = \langle\int_0^\cdot g(X_s)dX_s\rangle_t = \int_0^t g^2(X_s)d\langle X\rangle_s$

Theorem 1.20 (Quadratic covariation of 2 Itô integrals)**.** *Let* $g_1, g_2 \in \mathcal{C}^1$ *and define*

$$Y_t^j := \int_0^t g_j(X_s) dX_s,$$

for $j = 1, 2$, *then*

$$\left\langle Y^1, Y^2 \right\rangle_t = \int_0^t g_1(X_s) g_2(X_s) d\langle X \rangle_s.$$

Theorem 1.21. *Let* X *be as in Theorem 1.11,* $X = \left(X^j\right)_{j=1}^d = \left(A^1, \ldots, A^n, \Psi^1, \ldots, \Psi^m\right)$*, with* $n + m = d$ *and where* $A = \left(A^1, \ldots, A^n\right)$ *is of finite variation and* $\Psi = \left(\Psi^1, \ldots, \Psi^m\right)$ *is of continuous quadratic variation. Then for* $f \in \mathcal{C}^{1,2}(\mathbb{R}^n, \mathbb{R}^1)$*, we have*

$$\begin{aligned} f(A_t, \Psi_t) =& f(A_0, \Psi_0) + \int_0^t \nabla_a f(A_s, \Psi_s) dA_s + \int_0^t \nabla_\psi f(A_s, \Psi_s) d\Psi_s \\ &+ \frac{1}{2} \sum_{j,k=1}^m \int_0^t \frac{\partial f}{\partial \psi^j \partial \psi^k}(A_s, \Psi_s) d\langle \Psi^j, \Psi^k \rangle_s, \end{aligned} \tag{1.8}$$

where $\nabla_a f$ *and* $\nabla_\psi f$ *denote the gradients of* f *with respect to the* A *and* Ψ *coordinates respectively.*

Corollary 1.22. *For* $f \in \mathcal{C}^{1,2}(\mathbb{R}, \mathbb{R}^d)$*, and* X *continuous and of continuous quadratic variation, we have*

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s) ds + \int_0^t \nabla_x f(s, X_s) dX_s + \frac{1}{2} \sum_{j,k=1}^d \int_0^t \frac{\partial f}{\partial x^j \partial x^k}(s, X_s) d\langle X^j, X^k \rangle_s, \tag{1.9}$$

Theorem 1.23 (Associativity of the Itô integral)**.** *Let* X, f *be as in Theorem 1.11, and* Ψ *an* $\mathbb{R}^1$*-valued continuous function such that* $\int_0^t \Psi_s \nabla f(X_s) dX_s$ *exists. Then, the Itô integral* $\int_0^t \Psi_s d\tilde{X}_s$ *exists for* $\tilde{X}_t = \int_0^t \nabla f(X_s) dX_s$*, and*

$$\int_0^t \Psi_s d\tilde{X}_s \equiv \int_0^t \Psi_s d\left(\int_0^\cdot \nabla f(X_u) dX_u\right)_s = \int_0^t \Psi_s \nabla f(X_s) dX_s$$

As an application of the Itô formula, we solve the "stochastic differential equation" (SDE)

$$\begin{cases} S_0 &= 1 \\ dS_t &= rS_t dt + \sigma S_t dW_t, \end{cases} \tag{1.10}$$

where $r \in \mathbb{R}$, $\sigma > 0$, and W is a typical path of the Brownian motion ($t \mapsto W_t$ is continuous and $\langle W \rangle_t = t, \quad \forall t \geq 0$). In fact, (1.10) is an integral equation that reads

$$S_t = S_0 + \int_0^t S_s r ds + \int_0^t S_s \sigma dW_s, \quad \forall t \geq 0.$$

And the process

$$S_t := f(t, W_t) = e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}. \tag{1.11}$$

is straightforward to verify using the Itô formula to be a solution to (1.10).

2. The Idea of Perfect Dynamic Hedging and Valuation by No-Arbitrage

2.1 Motivating Example

Consider X_t the price of risky asset ("stock") at time $t \geq 0$. Assume that assets pay no dividends, and there is no storage cost. Consider a forward contract on the underlying, with maturity T (delivery date of the forward). Then, how should a short party (say a "bank") determine F (forward price), and hedge her risk after entering the contract? Classical valuation approach would be that the price of the forward is

$$\pi(H) = E\left[e^{-rT}\left(X_T - F\right)\right]. \tag{2.1}$$

Hence, $0 = \pi(H)$ yields $F = E\left[X_T\right]$. But this is wrong i.g., the only no-arbitrage price being $F - X_0 e^{rT}$. This can be shown by a replication of cashflows argument The references for this chapter are [Bjö98], [LL00] and [Shr04].

2.2 The Idea of Perfect Dynamic Hedging

2.2.1 Assumption

We assume that the interest rate is $r = 0$ (all prices are in discounted units). We denote by $X_t(\omega)$ the price at time t of the risky asset for a scenario $\omega \in \Omega$. We assume structurally that $t \mapsto X_t(\omega)$ is continuous and is of continuous quadratic variation $\langle X(\omega)\rangle_t = \int_0^t \left(\sigma(s, X_s(\omega))\right)^2 ds$, where $\sigma(t,x)$ is the "volatility profile". Moreover, we assume that the price evolution is unknown at time $t = 0$, and denote $X_0 = X_0(\omega)$.

Two examples of models satisfying these assumptions are given in the next section.

2.2.2 Model Examples

1. Bachelier Model (see [Bac00]).
   The stock price process is given via the SDE

   $$X_t = X_0 + mt + \sigma W_t, \tag{2.2}$$

   where m is the drift parameter, $\sigma > 0$ is the volatility parameter, W is a Brownian motion. From (2.2), we have

   $$\langle X\rangle_t = \langle \sigma W\rangle_t = \sigma^2 t, \quad t \geq 0, \quad \text{for a.a } \omega,$$

   that is $\sigma(t,x) = \sigma$ and is constant.

2. Black-Scholes Model.
   The stock price process is given via the SDE

   $$X_t = X_0 \exp\left(\left(m - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right), \tag{2.3}$$

which is a geometric Brownian motion (suggested by P. Samuelson as model for asset prices). The process X in (2.3) is the solution to the SDE

$$dX_t = X_t\left(mdt + \sigma dW_t\right), \qquad X_0 > 0,$$

from which we obtain

$$\langle X\rangle_t = \int_0^t \sigma^2 X_t^2 dt,$$

that is $\sigma(t,x) = \sigma x$.

Next, we discuss the no-arbitrage valuation of a derivative with payoff profile $H(\omega) = f(X_T(\omega))$, and also the possibility of perfect hedging. The approach on no-arbitrage valuation is to determine the fair price as the cost of the (dynamic) replication.

To this end, the ansatz is to solve the partial differential equation (PDE) for $F(t,x)$ (fair price)

$$\begin{cases} \frac{1}{2}\sigma^2 F_{xx} + F_t &= 0 \qquad (t,x) \in (0,T)\times\mathbb{R} \\ F(T,x) &= f(x) \end{cases} \tag{2.4}$$

Let $F \in \mathcal{C}\left([0,T]\times\mathbb{R}\right) \cap \mathcal{C}^2\left((0,T)\times\mathbb{R}\right)$ be solution[1] to (2.4).

Lemma 2.1. *Let F be as above, then it holds for all $\omega \in \Omega$, $t \in [0,T]$*

$$H(\omega) := f(X_T(\omega)) = F(0,X_0) + \int_0^T F_x(s, X_s(\omega))dX_s(\omega). \tag{2.5}$$

We observe:

- The fair price (at time $t = 0$) must be $F(0,X_0)$ to exclude arbitrage.
  Otherwise, the strategy

  a) Sell the contract at $t = 0$ for $\pi(H)$,
  b) trade according to $\vartheta_s(\omega) = F_x(s, X_s(\omega))$

  yields at time T: $-H(\omega) + \int_0^T \vartheta_s(\omega)dX_s(\omega) + \pi(H) = \pi(H) - F(0,X_0)$. This gives arbitrage profit if $\pi(H) - F(0,X_0) > 0$. In case $\pi(H) - F(0,X_0) < 0$, short the above strategy. So the only no-arbitrage price (NA-price) is $\pi(H) = F(0,X_0)$.

- Analogously, at any time $t \in [0,T]$ the replication cost for payoff H at T is $F(t, X_t(\omega))$.

2.3 Case Study: Bachelier Model

Our aim is to derive the solution to the PDE

$$\begin{cases} \mathcal{L}F := \frac{1}{2}\sigma^2 F_{xx} + F_t &= 0 \qquad (t,x) \in [0,T)\times\mathbb{R} \qquad \text{"dual heat equation"} \\ F(T,x) &= f(x). \end{cases} \tag{2.6}$$

[1] Note: $C^{1,2}$ would also suffice. Differentiability later actually shown on $[0,T)\times\mathbb{R}$.

The heat equation PDE

$$\frac{1}{2}F_{xx} - F_t = 0 \qquad (t,x) \in (0,\infty) \times \mathbb{R} \tag{2.7}$$

has fundamental solutions, for any $y \in \mathbb{R}$,

$$P(t,x,y) := P_t(x,y) := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right), \qquad x \in \mathbb{R}, t > 0. \tag{2.8}$$

For the boundary condition $F(T,x) = f(x)$, where we assume that f is continuous and satisfies the growth condition

$$|f(y)| \leq C\exp(C|x|^\alpha) \tag{2.9}$$

with $C \in [1,\infty)$ and $\alpha \in (0,2)$, we then let

$$F(t,x) := \int_{\mathbb{R}} f(y) P_{\sigma^2(T-t)}(x,y)dy, \qquad (t,x) \in [0,T) \times \mathbb{R}, \tag{2.10}$$

and have

$$f(x) = F(T,x) = \lim_{t\uparrow T} F(t,x).$$

One can check that F defined in (2.10) solves the PDE (2.6) since the integrand function does so. To justify that rigorously, one has to justify to differentiate under the integral, arguing with limits of difference quotients and dominated converge, in order to justify the interchange of the limit operations of partial derivation ∂ and integration $\int$ in (2.10).

The expression of F in (2.10) can be written as

$$\begin{aligned} F(t,x) &= \int_{\mathbb{R}} f(y) \frac{1}{\sigma\sqrt{2\pi(T-t)}} \exp\left(-\frac{(x-y)^2}{2\sigma^2(T-t)}\right)dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x+\sigma z\sqrt{T-t}) \exp\left(-\frac{z^2}{2}\right)dz \\ &= \tilde{E}\left[f(x+\sigma\sqrt{T-t}Z)\right], \end{aligned} \tag{2.11}$$

for $Z \sim \mathcal{N}(0,1)$ under $\tilde{P}$ (on some other probability space).

Stochastic Interpretation

Recall that one writes $d\tilde{P} = ZdP$ if $\tilde{E}[H] = E[ZH]$ for all $H = 1_A, A \in \mathcal{F}$ (or, equivalently, for all $\mathcal{F}$-measurable $H \geq 0$, or all $H \in L^(\mathcal{F})$); cf. the notion of Radon Nikodym derivative.

Lemma 2.4 (Bayes' Formula). *For $\tilde{P} \sim P$ on $(\Omega,\mathcal{F})$, $\mathcal{F}_t \subseteq \mathcal{F}$, and $Z := \frac{d\tilde{P}}{dP}\Big|_{\mathcal{F}} > 0$ a.s. then,*

$$H_t := \tilde{E}_t[H] = \frac{E_t[HZ]}{E_t[Z]}, \qquad \text{for } H \geq 0 \text{ or } HZ \in L^1(P), \tag{2.12}$$

where $E_t[.] = E[.|\mathcal{F}_t] = E^P[.|\mathcal{F}_t]$ denotes the conditional P-expectation given $\mathcal{F}_t$.

Theorem 2.5 (Change of measure on Wiener space, special case of the Cameron-Martin-Girsanov theorem)**.** *Let $W = (W_t)$, $t \in [0,T]$, be an $(\mathcal{F}_t)$-Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\le T}, P)$. Let[2] $\mathcal{F} = \mathcal{F}_T$. Then the following statements hold:*

i) There is an equivalent probability measure $P^* \sim P$ such that $W_t^* = W_t + \frac{m}{\sigma}t$, $t \in [0,T]$, is a P^*-Brownian motion,

ii) $P^* \sim P$ has the density $Z := \frac{dP^*}{dP}\big|_{\mathcal{F}_T} = \exp\left(\alpha W_T - \frac{1}{2}\alpha^2 T\right)$ for $\alpha = -\frac{m}{\sigma}$.

Hence, we can write the NA-price in (2.11) in the Bachelier model as

$$\begin{aligned} F(t, X_t(\omega)) &= E^*\left[f\left(x + \sigma(W_T^* - W_t^*)\right)\right]\big|_{x = X_t(\omega)} \\ &= E_t^*\left[f\left(X_t + \sigma(W_T^* - W_t^*)\right)\right] \\ &= E_t^*\left[f\left(X_t + (X_T - X_t)\right)\right] \\ &= E_t^*\left[f\left(X_T\right)\right], \end{aligned} \tag{2.13}$$

Theorem 2.7 (Risk neutral valuation in the Bachelier model)**.**

a) Under P^*, X_t and $\pi_t := F(t, X_t)$ are martingales,

b) The NA-price process π_t of the derivative satisfies

$$\pi_t(f(X_T)) = E_t^*\left[f\left(X_T\right)\right], \qquad \forall t \le T. \tag{2.14}$$

Corollary 2.8 (Put-Call parity)**.** *The prices (at time t) of (European) call and put options on the same underlying, same maturity T and same strike K must satisfy*

$$\pi_t(\text{Call}) = \pi_t(\text{Put}) + X_t - K \qquad \forall t \in [0,T]. \tag{2.15}$$

Exercise: By 1-period arbitrage arguments one can show more generally that a similar Put-Call parity like (2.15) holds true even model independently, if X_t is replaced by the forward price of the underlying and K by the zero coupon bond price at t (for maturity T).

2.4 Case Study: Black-Scholes Model

Similar to what we have done in Section 2.3, we look for the solution $F \in \mathcal{C}\left([0,T] \times \mathbb{R}_+\right) \cap \mathcal{C}^2\left((0,T) \times \mathbb{R}_+\right)$ to the PDE

$$\begin{cases} \frac{1}{2}\sigma^2 x^2 F_{xx} + F_t &= 0 \qquad (t,x) \in [0,T) \times \mathbb{R}_+ \\ \qquad\quad F(T,x) &= f(x). \end{cases} \tag{2.16}$$

The solution to the PDE (2.16) is obtained from that of the PDE (2.6) as

$$\begin{aligned} F(t,x) &= G\left(t, \log x - \frac{1}{2}\sigma^2(T-t)\right) \\ &= \int_{\mathbb{R}} f\left(x \exp\left(y - \frac{1}{2}\sigma^2(T-t)\right)\right) \frac{1}{\sigma\sqrt{2\pi(T-t)}} \exp\left(-\frac{y^2}{2\sigma^2(T-t)}\right) dy \end{aligned}$$

[2] w.l.o.g.

where G is the solution to the PDE (2.6) with boundary condition $g(x) = f(e^x)$. In this case, a sufficient growth condition for f is $|f(x)| \leq C(x^{-n} + x^n)$ with $C \in \mathbb{R}$ and $n \in \mathbb{N}$; F can be rewritten as

$$F(t,x) = \int_{\mathbb{R}} f\left(x \exp\left(z\sigma\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right)\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz. \tag{2.17}$$

Theorem 2.10. *The function F defined as in (2.17) solves the PDE (2.16).*

Theorem 2.11 (Stochastic interpretation, risk neutral valuation)**.** *Under the "risk neutral" measure P^* defined via*

$$\frac{dP^*}{dP} = \exp\left(-\frac{m}{\sigma}W_T - \frac{1}{2}\frac{m^2}{\sigma^2}T\right),$$

it holds that

i) the process W_t^* $(t \in [0,T])$ is a P^*-Brownian motion, and

ii) the price process $(X_t)_{t \leq T}$ of the underlying and the no-arbitrage price process $(\pi_t)_{t \leq T} = (F(t,X_t))_{t \leq T}$ of the derivative are martingales under P^*, satisfying

$$X_t = E_t^*[X_T] \quad \text{and} \quad F(t,X_t) = E_t^*[f(X_T)] \qquad \forall t \leq T.$$

2.5 The Idea of Perfect Hedging with Non-Zero Interest Rates

2.5.1 Model

Our model in this section comprises two traded assets. Unlike in the previous section, when $r \neq 0$ the price of the riskless asset, the savings account, however is not constant with value 1 but varies over time, albeit in a deterministic exponential fashion.

1) The "risky asset" (e.g. stock) with price process $(S_t(\omega))_{t \leq T}$, such that $t \mapsto S_t(\omega)$ is continuous, of continuous quadratic variation with $\langle S(\omega)\rangle_t = \int_0^t \sigma^2(u, S_u(\omega))du$, $t \leq T$, where $\sigma(t,x)$ is the volatility profile.

2) The savings account with some (compound) interest rate r for borrowing and lending, with price at time t, $B_t = \exp(rt)$.

Remark 2.12. • *The value of 1 Euro at time T at some earlier time $t \leq T$ (maturity T zero coupon bond price at t) is : $\exp(-r(T-t)) = \frac{B_t}{B_T}0 =: B_{t,T}$.*

The derivatives of European type payoff at time T are

$$\tilde{H}(\omega) = f(S_T(\omega)) = H(\omega)e^{rT}, \tag{2.18}$$

where $\tilde{H}$ denote the discounted payoff, and H the non discounted payoff.

Definition 2.13 (Dynamic trading strategy)**.** *A dynamic trading strategy with respect to assets $\begin{pmatrix} B \\ S \end{pmatrix}$ is a couple $\begin{pmatrix} \eta \\ \vartheta \end{pmatrix} = \left(\begin{pmatrix} \eta_t \\ \vartheta_t \end{pmatrix}\right)_{t \leq T}$ with suitable conditions such that $\int_0^\cdot \eta dB + \int_0^\cdot \vartheta dS$ is well-defined.*

Definition 2.14. *The trading strategy* $\begin{pmatrix} \eta \\ \vartheta \end{pmatrix}$ *is called a self-financing (s.f.) strategy if the wealth process defined by*

$$\tilde{V}_t = \eta_t B_t + \vartheta_t S_t$$

satisfies

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \eta_s dB_s + \int_0^t \vartheta_s dS_s, \qquad \forall t \leq T.$$

Lemma 2.16. *A strategy* $\begin{pmatrix} \eta \\ \vartheta \end{pmatrix}$ *is self-financing with respect to* $\begin{pmatrix} B \\ S \end{pmatrix}$ *if and only if it is self-financing with respect to* $\begin{pmatrix} 1 \\ X \end{pmatrix} = \frac{1}{B}\begin{pmatrix} B \\ S \end{pmatrix}$, *with* $X = \frac{S}{B}$.

2.5.2 Construction of Hedging Strategy

For $\tilde{H} = g(S_T)$, assume that $\{S_u(\omega) \mid u \leq T\} \subset \mathbb{R}$ or $\mathbb{R}_+$, write $\mathbb{R}_{(+)}$.

Theorem 2.17. *Let* $G(t,x) \in \mathcal{C}\left([0,T] \times \mathbb{R}_{(+)}\right) \cap \mathcal{C}^2\left((0,T) \times \mathbb{R}_{(+)}\right)$ *(or more generally with just* $G(t,x) \in \mathcal{C}^{1,2}\left((0,T) \times \mathbb{R}_{(+)}\right)$ *being such that* $t \mapsto G(t, S_t(\omega))$ *is*[3] *continuous on* $[0,T]$ *for a.a.*ω*) such that it solves the Cauchy problem*

$$\begin{cases} \frac{1}{2}\sigma^2(.,.)G_{xx} + G_t + rxG_x - rG &= 0 \qquad \forall (t,x) \in (0,T) \times \mathbb{R}_{(+)} \\ G(T,x) &= g(x) \quad \forall x \in \mathbb{R}_{(+)}, \end{cases} \tag{2.19}$$

then

$$\tilde{H}(\omega) = g(S_T(\omega)) = G(t, S_t(\omega)) + \int_t^T \eta_u(\omega) dB_u(\omega) + \int_t^T \vartheta_u(\omega) dS_u(\omega) \tag{2.20}$$

for a self financing strategy $\begin{pmatrix} \eta \\ \vartheta \end{pmatrix}$ *given by*

$$\vartheta_t(\omega) = \vartheta_t(t, S_t(\omega)) = G_x(t, S_t(\omega)), \tag{2.21}$$

$$\eta_t(\omega) = \eta_t(t, S_t(\omega)) = \frac{G - xG_x}{B_t}(t, S_t(\omega)). \tag{2.22}$$

that satisfies $dG(t, S_t(\omega)) = \eta_u(\omega) dB_u(\omega) + \vartheta_u(\omega) dS_u(\omega)$,

2.5.3 Case study: Black-Scholes Model with Non-Zero Interest Rate ($r \in \mathbb{R}$)

The model is for $S_t = B_t X_t$ with (X_t) from the (zero-interest) BS.model of section 2.4. The pricing and hedging problem corresponds to solving the Cauchy problem

$$\begin{cases} \frac{1}{2}\sigma^2 x^2 G_{xx} + G_t + rxG_x - rG &= 0 \qquad \forall (t,x) \in (0,T) \times \mathbb{R}_+ \\ G(T,x) &= g(x) \quad \forall x \in \mathbb{R}_+. \end{cases} \tag{2.23}$$

Theorem 2.18. *The solution to the Cauchy Problem (PDE) (2.23) is given by*

$$G(t,x) = F(t, xe^{-rt})e^{rt}, \tag{2.24}$$

where $F(t,x)$ *is the solution to the PDE (2.16) with boundary condition* $g(x) = e^{rT} f(xe^{-rT})$*, respectively* $f(y) = g(ye^{rT})e^{-rT}$.

[3] instead of $G(t,x) \in \mathcal{C}\left([0,T] \times \mathbb{R}_{(+)}\right)$

Stochastic Interpretation

Theorem 2.19.

a) Under P^*, $X_t := \frac{S_t}{B_t}$ is a martingale.

b) For payoff function $g(.)$ (at most of polynomial growth), the NA-price of the derivative is

$$G(t, S_t) = B_t E_t^* \left[\frac{1}{B_T} g(S_T) \right], \quad \forall t \leq T, \tag{2.25}$$

and $\frac{G(t,S_t)}{B_t}$ is a P^*-martingale in particular.

Corollary 2.21. *For $g(x) = (x - K)^+$ (call option) in the Black-Scholes model, the no arbitrage price is given by the Black-Scholes formula*

$$\pi_t(\text{Call}) = S_t \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-) \tag{2.26}$$

$$= B_{t,T} \left(F_t \Phi(d_+) - K \Phi(d_-) \right), \tag{2.27}$$

with $d_\pm := \dfrac{\log\left(\frac{S_t}{K}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = \dfrac{\log\left(\frac{F_t}{K}\right) \pm \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{T-t}}$. Variant (2.27) is the convenient "Black's formula".

Discussion of the Black-Scholes Model assumptions

- Model assumes frictionsless market (no transaction costs, no taxes, no restrictions on short sales, same interest rate for borrowning and lending, no (adverse) market impact of trades...) - idealization of reality.
- Black Scholes model is used as a standard reference model to quote option prices: A well known saying is "The implied volatility is the wrong number to be put in the wrong model to give the right price!".
- In reality, hedging must be done in discrete time, e.g. discrete approximation by Δ_{t_k} risky assets at time t_k, hold over (t_k, t_{k+1}) and re-hedge at time t_{k+1}. One could investigate the resulting hedging-errors from discrete approximation by simulation, to actually see e.g. that

  a) trading according to a time-discrete implementation of the (continuous) Δ-("delta")-hedging strategy (i.e. with portfolio adjustments only being done at discrete time points) leads to a diminishing hedging error for refining time grids; b) When implemented in discrete time, more advanced strategies like Δ/Γ-hedging (see e.g. [Hul06]) can improve on the performance of plain Δ-hedging, i.e. obtain smaller hedging errors.

2.5.4 Black-Scholes Models With Dividends, and FX-options

Dividends Payed Continously

Here, we denote by A_t the nominal risky asset ("stock") price at time t, such that

$$A_t = A_0 \exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2 \right) t \right), \quad \forall t \leq T. \tag{2.28}$$

The riskless asset price is, as in Section 2.5.1, $B_t = e^{rt}$. Assume dividends are payed continuously according to some dividend rate $\delta \geq 0$ as fractions of the current risky asset value.

Then $S_t = a_t A_t = A_0 \exp\left(\sigma W_t + \left(\mu + \delta - \frac{1}{2}\sigma^2\right) t\right)$ is the value process of a self-financing strategy, that re-invests all dividends into stock, with $A_0 = S_0 = X_0$. Moreover,

$$\begin{aligned} X_t &= \frac{S_t}{B_t} \\ &= e^{(\delta - r)t} A_t = S_0 \exp\left(\sigma W_t^* - \frac{1}{2}\sigma^2 t\right), \end{aligned}$$

with W^* a P^*-Brownian motion, where $\frac{dP^*}{dP} = \exp\left(-\frac{m}{\sigma} W_T - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right)$ on $\mathcal{F}_T$ for $m = \mu + \delta - r$. Therefore, we can proceed as before, by considering S_t as the (non-discounted) value process of a tradable risky asset. A few example of how we can proceed follow.

Examples

1. The forward price F_t at time $t \leq T$, on A_T at T is

$$F_t = e^{(r-\delta)T} X_t = e^{\tilde{r}(T-t)} A_t, \tag{2.29}$$

   where $\tilde{r} = r - \delta$.

   Exercise: One can also derive (2.29) by a direct static replication argument.

2. Consider the payoff $H = (A_T - K)^+ = e^{-\delta T}\left(S_T - \tilde{K}\right)^+$, with $\tilde{K} = e^{\delta T} K$. Then the general so-called "Black's Version" of the Black-Scholes formula for pricing European call options in a Black-Scholes model with or without dividends is

$$\pi_t = B_{t,T}\left(F_t \Phi(d_+) - K\Phi(d_-)\right), \tag{2.30}$$

   with $d_\pm := \dfrac{\log\left(\frac{F_t}{K}\right) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$.

Note that formulae (2.30) and (2.27) are identical.

Remark 2.25. *For discretly payed dividends, see alternative models e.g. in [KR05].*

Foreign Exchange Derivatives

Similar arguments as for dividends can be used for FX-options (Foreign eXchange options on exchange rates of currencies). The model is the following

- $B_t^d = e^{r^d t}$ (in Euro) is the domestic savings account,
- $B_t^f = e^{r^f t}$ (in Dollar) is the foreign savings account,
- $\chi_t = \chi_0 \exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right)$; $(\sigma > 0)$ is the FX rate quote of one Dollar in Euro (quoted as "\$/EUR"). It is not a tradable asset.

Consider an option with payoff $H = (\chi_T - K)^+$. One can consider r^f like a dividend rate δ of an tradable but risky asset $S_t = B_t^f \chi_t$. The NA-price of a derivative with payoff $H = (\chi_T - K)^+$ is then obtained as (apply (2.30) by

$$\pi_t = e^{-r^f(T-t)} \chi_t \Phi(d_+) - K e^{-r^d(T-t)} \Phi(d_-).$$

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