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De Casteljau's algorithm is an extrapolation method

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Abstract

One of the most important recursive schemes in CAGD is De Casteljau's algorithm for the evaluation of Bézier curves and surfaces. Within the theory of triangular recursive schemes we discuss the De Casteljau's algorithm as a particular case, i.e. we prove that it is identical to the E-algorithm (or GNA-algorithm) in a particular frame. This result is of theoretical interest since it leads to some classification of recurrence relations in CAGD. Furthermore, it may be regarded as a model example how to obtain known and possibly new recursive schemes in CAGD as examples of the theory of general extrapolation algorithms.

Keywords: CAGD; Recurrence scheme; De Casteljau's algorithm; Bernstein polynomials; Extrapolation algorithms; E-algorithm; GNA-algorithm

1. Introduction

In computer aided geometric design (CAGD) one is interested in representations of curves and surfaces which allow a fast evaluation in order to obtain a fast plot of the object under consideration. This motivated the development of triangular recursion schemes in the past decades, such as De Casteljau's algorithm, De Boor's algorithm, and Goldman's algorithm, among others (Farin, 1990). The general structure of such triangular recursions was studied in (Brezinski, 1980; Mühlbach, 1978) generalizing the classical Neville–Aitken algorithm.

It is of some theoretical interest to classify the above schemes in CAGD within the general frame of extrapolation algorithms. Thus, known algorithms can be characterized

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and possibly new algorithms can be found. In this work, we treat De Casteljau's algorithm as a model example and prove that it is indeed a particular case of the E-algorithm (Brezinski, 1980) or GNA-algorithm (Mühlbach, 1978). There a point of a Bézier curve is characterized as a solution of an interpolation problem and the Bernstein polynomials are characterized as solutions of a certain system of linear equations.

For convenient reading we recall the notations and the general scheme of the E-algorithm in Section 2. Then, we state and prove in Section 3 that the E-algorithm reduces to De Casteljau's algorithm in a certain setting. In (Brezinski and Walz, 1991) a general frame was given for such triangular recursions and De Casteljau's algorithm was considered as one particular case. In order to show that our result does not contradict (Brezinski and Walz, 1991), we recall the definitions of the representation functional and the characteristic space in Section 4. Then we prove in Section 5 that De Casteljau's algorithm can be treated in the context of (Brezinski and Walz, 1991).

2. General extrapolation algorithms

In the context of linear or quasi-linear extrapolation one is concerned with a system of linear equations (yielded by the interpolation conditions) such that a certain approximant E_n^k is given as a quotient of determinants (see (Brezinski, 1980; Mühlbach, 1978)). Using the notations of the so-called E-algorithm we have the data

$$g_k(j) \in \mathbb{K} \quad (k, j = 0, 1, 2, \ldots, n) \tag{1}$$

and

$$E_j^0 \in E \quad (j = 0, 1, 2, \ldots, n)$$

where E is a vector space over $\mathbb{K}$, $\mathbb{K}$ being the real or complex field. We always assume that the data satisfy

$$0 \neq |g_0(j), \ldots, g_k(j)| := \det(g_i(\ell))_{i=0,\ldots,k}^{\ell=j,\ldots,j+k} \tag{2}$$

for all $k = 0, \ldots, n$ and $j = 0, \ldots, n-k$. Then, for any $k = 1, \ldots, n$ and $j = 0, \ldots, n-k$,

$$E_j^k := \frac{|E_j^0, g_1(j), \ldots, g_k(j)|}{|g_0(j), \ldots, g_k(j)|} \in E \tag{3}$$

where the numerator determinant is defined in (2) and the denominator determinant

$$|E_j^0, g_1(j), \ldots, g_k(j)| := \det \begin{pmatrix} E_j^0 & \ldots & E_{j+k}^0 \\ g_1(j) & \ldots & g_1(j+k) \\ \vdots & & \vdots \\ g_k(j) & \ldots & g_k(j+k) \end{pmatrix} \in E$$

is defined by formal expansion of the determinant with respect to the first row.

Remark 1. In view of $g_0(\ell) \neq 0$ (cf. (2)) with no loss of generality we may and we do assume, for convenience, that $g_0(\ell) = 1$ for all $\ell = 0, \ldots, n$ (Brezinski, 1980).

Indeed, by dividing the corresponding rows of the numerator and denominator in (3) by $g_0(\ell)$ this may be assumed simply by changing the input data E_ℓ^0 to $E_\ell^0/g_0(\ell)$. Under this assumption (3) holds also for $k=0$ and $j=0,\dots,n-k$.

Remark 2. Assuming $E_0^0,\dots,E_n^0$ linearly independent define linear functionals D_i $(i=0,\dots,n)$ on $\operatorname{span}\{E_0^0,\dots,E_n^0\}$ by

$$D_i(E_j^0) := g_i(j) \quad (i,j=0,\dots,n).$$

Then for any $k=0,\dots,n$ and $j=0,\dots,n-k$ E_j^k as defined in (3) is the unique linear combination of $E_j^0,\dots,E_{j+k}^0$ satisfying the *interpolation conditions*

$$D_i(E_j^k) := \delta_{i,0} \quad (i=0,\dots,k). \tag{4}$$

This is obvious from (3). Hence, once we have shown that the entries in De Casteljau's algorithm are given in terms like (3) we have proved that De Casteljau's algorithm is an extrapolation method and the Bézier curves are interpolants in a certain extrapolation scheme.

This is a different point of view than the usual one. Usually, given control points $E_0^0,\dots,E_n^0$ and a parameter $t\in(0,1)$ De Casteljau's algorithm computes the point

$$E_0^n=\sum_{j=0}^{n}\binom{n}{j}\cdot t^j\cdot(1-t)^{n-j}\cdot E_j^0=(1-t)\cdot E_0^{n-1}+t\cdot E_1^{n-1}$$

of the Bézier curve by repeated linear interpolation with weights $1-t$ and t, respectively. Knowing that the entries E_j^k of De Casteljau's algorithm are of the form (3) then from the general frame of linear extrapolation theory we know that they are uniquely defined by the interpolation conditions (4). Then, moreover, as a consequence De Casteljau's algorithm follows as will be shown below.

Remark 3. From the general extrapolation theory (Brezinski, 1980; Mühlbach, 1978, 1981) it is known that under the assumptions (2) all elements (3) can be computed by the recursive triangle

$$E_j^k=(1-\gamma_j^k)\cdot E_j^{k-1}+\gamma_j^k\cdot E_{j+1}^{k-1} \tag{5}$$

for all $k=0,\dots,n$ and $j=0,\dots,n-k$ where the coefficients $\gamma_j^k\in\mathbb{K}$ are determined by

$$\gamma_j^k=\frac{g_{k-1,k}(j)}{g_{k-1,k}(j)-g_{k-1,k}(j+1)} \tag{6}$$

provided they all are well defined, i.e. their denominators are nonzero. It is important that the entries $g_{k,j}(\nu)$ themselves can be computed recursively according to (5)

$$g_{k,j}(\nu)=(1-\gamma_\nu^k)\cdot g_{k-1,j}(\nu)+\gamma_\nu^k\cdot g_{k-1,j}(\nu+1) \tag{7}$$

starting from $g_{0,j}(\nu)=g_j(\nu)/g_0(\nu)$. We refer to (Mühlbach, 1978, 1981; Brezinski, 1980) for proofs, examples, applications, and further properties of such general extrapolation schemes.

Remark 4. Note that the computational costs are of third order in n while the computation of E_j^k for all $k = 0, \ldots, n$ and $j = 0, \ldots, n-k$ via Gaussian elimination is of fourth order. But the crucial point compared with the fast algorithms used in CAGD is that there the coefficients γ_j^k are known explicitly. Then, the computer effort for the computation of E_j^k (for all $k = 0, \ldots, n$ and $j = 0, \ldots, n-k$) is only of quadratic order in n. We are going to show that De Casteljau's algorithm can be derived as a special case from this general recursive scheme.

From the general framework of extrapolation methods we recall that $g_{k,j}(\nu)$ is again a quotient of determinants,

$$g_{k,j}(\nu) = \frac{|g_j(\nu), g_1(\nu), \ldots, g_k(\nu)|}{|g_0(\nu), \ldots, g_k(\nu)|}, \tag{8}$$

(compare with (3)).

The classical example yielding an algorithm of quadratic order is the Neville–Aitken algorithm where $g_k(j) = (x_j)^k$ is the kth power of the node x_j. Then, the determinants in (8) are known explicitly.

3. De Casteljau's algorithm is an extrapolation algorithm

In this section we consider De Casteljau's algorithm which computes the value

$$T_n^n := \sum_{i=0}^{n} B_i^n(t) \cdot P_i \tag{9}$$

of a Bézier curve for the parameter $t \in (0,1)$ and the control points $P_0, \ldots, P_n \in E$ via

$$T_i^k = t \cdot T_i^{k-1} + (1-t) \cdot T_{i-1}^{k-1} \tag{10}$$

where $T_i^0 := P_i$ and $B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i}$ is a Bernstein polynomial.

Since (10) has a structure similar to (5) one should expect that the general extrapolation algorithm (5) includes De Casteljau's algorithm (10) as a particular case.

In order to prove this we have to find data as in (1) satisfying

(a) (2),
(b) $E_0^n = T_n^n$ where E_0^n is defined by (3) and T_n^n is defined by (9) with $E_j^0 = P_j$, and
(c) $\gamma_j^k = t$ for all $k = 0, \ldots, n$ and $j = 0, \ldots, n-k$ where γ_j^k is defined by (6) and (8).

Starting with $g_0(j) = 1$ one constructs $g_1(j)$ such that $\gamma_j^1 = t$. Then, via mathematical induction on $k = 2, 3, \ldots, n$ one tries to find $g_k(j)$ satisfying $\gamma_j^k = t$. Inspecting the determinants one observes that once $g_0(j), \ldots, g_{k-1}(j)$ are known one has k free values to chose $g_k(j)$. Choosing $0 = g_k(0) = \ldots = g_k(k-2)$ and $1 = g_k(k-1)$ one obtains the following result.

Theorem 1. *Let $g_0(j) = 1$ for any $j = 0,\dots,n$ and let, for $k = 1,\dots,n$ and $j = 0,\dots,n$,*

$$g_k(j) = \begin{cases} 0 & \text{if } j \leqslant k-2, \\ \binom{j}{k-1}(1-1/t)^{j-k+1} & \text{if } j \geqslant k-1. \end{cases} \tag{11}$$

Then, the general extrapolation algorithm (5) *is De Casteljau's algorithm, i.e. we have* (a), (b), *and* (c) *such that, for $E_j^0 = P_j$, $j = 0,\dots,n$,* (5) *is* (10).

Remark 5. Although we consider only a finite part of the extrapolation table, i.e. n is a fixed bound for all indices, note that the theorem holds for all n simultaneously, i.e. for the full (infinite) extrapolation table.

In the proof of the theorem we need the following rule where we use the convention $\binom{j}{k} := 0$ for $j < k$.

Lemma 1. *For $\tau \in \mathbb{K} \setminus \{0\}$ and $n, k = 0, 1, 2, \dots$ the polynomial*

$$\begin{vmatrix} 1 & x & \dots & x^k \\ \binom{n}{0}\tau^n & \binom{n+1}{0}\tau^{n+1} & \dots & \binom{n+k}{0}\tau^{n+k} \\ \vdots & \vdots & & \vdots \\ \binom{n}{k-1}\tau^{n-k+1} & \binom{n+1}{k-1}\tau^{n-k+2} & \dots & \binom{n+k}{k-1}\tau^{n+1} \end{vmatrix} \tag{12}$$

in x is equal to

$$\tau^{k(n+1)} \cdot (1 - x/\tau)^k. \tag{13}$$

Proof of the lemma. Let us denote a generalized Vandermonde determinant with possibly repeated nodes by

$$V\left|{}^{f_0,\dots,f_n}_{\tau_0,\dots,\tau_n}\right| = \det\left(\left(\frac{d}{dx}\right)^{\mu_j(\tau_j)} f_i(\tau_j)\right)_{j=0,\dots,n}^{i=0,\dots,n}.$$

Here $\mu_j(\tau)$ is the multiplicity of τ in $(\tau_0,\dots,\tau_{j-1})$, and we are going to take $f_i(x) = p_i(x) := x^i$. Observe that the polynomial (12) equals

$$\frac{1}{x^n}\frac{1}{1!\cdot 2!\cdots(k-1)!} \cdot V\left|{}^{p_n,p_{n+1},\dots,p_{n+k}}_{x,\tau,\dots,\tau}\right|.$$

Being a linear combination of x^n, ..., x^{n+k} and having the k-fold zero τ, $\frac{1}{1!\cdot 2!\cdots(k-1)!} \cdot V\left|{}^{p_n,p_{n+1},\dots,p_{n+k}}_{x,\tau,\dots,\tau}\right|$ must equal $a_{n,k}(\tau) \cdot x^n \cdot (\tau - x)^k$ where the coefficient of x^n is

$$a_{n,k}(\tau)\cdot\tau^k=\begin{vmatrix}\binom{n+1}{0}\tau^{n+1} & \cdots & \binom{n+k}{0}\tau^{n+k}\\ \vdots & & \vdots\\ \binom{n+1}{k-1}\tau^{n-k+2} & \cdots & \binom{n+k}{k-1}\tau^{n+1}\end{vmatrix}$$
$$=\frac{1}{1!\cdot 2!\cdots(k-1)!}\cdot V\left|\begin{matrix}p_{n+1},\ldots,p_{n+k}\\ \tau,\ldots,\tau\end{matrix}\right|.$$

By multiplying the jth row by τ^{j-1} it is possible to take out of the ith column the factor τ^{n+i}, i.e. we have $a_{n,k}(\tau)\cdot\tau^k=\tau^{k(n+1)}\cdot b_{n,k}$ where

$$b_{n,k}=\begin{vmatrix}\binom{n+1}{0} & \cdots & \binom{n+k}{0}\\ \vdots & & \vdots\\ \binom{n+1}{k-1} & \cdots & \binom{n+k}{k-1}\end{vmatrix}.$$

From Pascal's triangle $\binom{\alpha+1}{k+1}-\binom{\alpha}{k+1}=\binom{\alpha}{k}$ one infers that $b_{n,k}=b_{n,k-1}=\cdots=b_{n,1}=1$ is independent of n and k. □

Proof of the theorem. Let $\tau:=1-1/t$ so that $g_k(j)=\binom{j}{k-1}\tau^{j-k+1}$ for all $k,j\geqslant 0$ (using the convention $\binom{j}{k}:=0$ for $j<k$). Then, from Lemma 1 we obtain

$$|g_0(n),\ldots,g_k(n)|=\tau^{k\cdot n}\cdot(\tau-1)^k \tag{14}$$

which is nonzero and proves (a).

The coefficient of E_j^0 in the numerator determinant of E_0^n (as defined in (3)) equals the coefficient of x^j in (12) written for $n=0$ and $k=n$. Hence, by Lemma 1 and (14) also written for $n=0$ and $k=n$, we obtain that the coefficient of E_j^0 in E_0^n equals

$$\frac{\tau^{n-j}(-1)^j\binom{n}{j}}{(\tau-1)^n}=B_j^n(t)$$

which proves $E_0^n=T_n^n$ (cf. (9)), i.e. (b).

Consider the coefficient of x^k in (12) and (13) to verify

$$|g_1(\nu),\ldots,g_k(\nu)|=\tau^{\nu\cdot k}.$$

Using this and (14) in (8) shows

$$g_{k-1,k}(\nu)=\frac{\tau^\nu}{(1-\tau)^{k-1}}.$$

Hence (6) leads to $\gamma_j^k=1/(1-\tau)=t$ which is (c). □

Remark 6. According to remark 2 from the general extrapolation theory we get the following interpretation of the point

$$T_n^n = E_0^n = \sum_{j=0}^{n} B_j^n(t) \cdot P_j$$

of a Bézier curve corresponding to the parameter $t \in (0,1)$. Defining linear functionals D_i $(i = 0,\dots,n)$ on $E_n := \operatorname{span}\{P_0,\dots,P_n\} \subset E$ by $D_0(P_j) = g_0(j) = 1$ $(j = 0,\dots,n)$ and for $i = 1,\dots,n$

$$D_i(P_j) = g_i(j) = \binom{j}{i-1} \cdot \left(1 - \frac{1}{t}\right)^{j-i+1} \quad (j = 0,\dots,n)$$

where we assume $\dim E_n = n+1$ then T_n^n is the unique element of E_n satisfying

$$D_i(T_n^n) = \delta_{i,0} \quad (i = 0,\dots,n).$$

This means that the Bernstein polynomials $B_j^n(t)$ are uniquely determined as solution of the system of linear equations

$$\sum_{j=0}^{n} B_j^n(t) = 1,$$

$$\sum_{j=0}^{n} B_j^n(t) \cdot \binom{j}{i-1} \cdot \left(1 - \frac{1}{t}\right)^{j-i+1} = 0 \quad (i = 1,\dots,n).$$

Inserting $B_j^n(t) = \binom{n}{j} t^j (1-t)^{n-j}$ in the last equations yields the identities

$$\sum_{j=0}^{n} \binom{n}{j} \cdot \binom{j}{i-1} \cdot (-1)^j = 0 \quad \text{for } i = 1,\dots,n$$

which, we suppose, are familiar in combinatorics.

4. Triangular recursions, reference functionals and characteristic spaces

In this section we recall the definitions of a reference functional and of a characteristic space (Brezinski and Walz, 1991) of a linear sequence transformation.

In the previous sections we considered a finite number of coefficients (cf. e.g. (10) where n is an upper bound of all indices). Following (Brezinski and Walz, 1991) we now consider sequence transformations of the type

$$T_j^k := \sum_{\nu=j-k}^{j} \beta_{j,\nu}^k \cdot T_\nu^0 \in E \tag{15}$$

where $(T_\nu^0)_{\nu\in\mathbb{Z}}$ is a sequence in E, E is a vector space over $\mathbb{K}$, and $\beta_{j,\nu}^k \in \mathbb{K}$ for all $j, \nu \in \mathbb{Z}$, $k = 0,1,2,\dots$.

Definition 1. Let F be a vector space over $\mathbb{K}$ with algebraic dual F^*. Let $(f_\nu)_{\nu=0,1,2,\ldots}$ be a sequence in F and $(L_\nu)_{\nu\in\mathbb{Z}}$ be a (bi-infinite) sequence in F^*. Then, for $j \in \mathbb{Z}$ and $k = 0, 1, 2, \ldots,$

$$R_j^k := \sum_{\nu=j-k}^{j} \beta_{j,\nu}^k \cdot L_\nu \in F^* \tag{16}$$

is the *reference functional* of (15) with respect to $(L_{j-k}, \ldots, L_j)$ if $L_{j-k}, \ldots, L_j$ are linearly independent. A $(k+1)$-dimensional subspace $E_k = \operatorname{span}\{f_0, \ldots, f_k\}$ of E is a *characteristic space* of the reference functional (16) if

$$\omega_\ell^m := R_\ell^m f_0 \neq 0 = R_\ell^m f_1 = \ldots = R_\ell^m f_m \tag{17}$$

for all $m = 0, \ldots, k$ and $\ell = j - k + m, \ldots, j$.

A sufficient condition for the transformation (15) to result from a triangular recurrence

$$T_\ell^m = \mu_\ell^m \cdot T_\ell^{m-1} + \lambda_\ell^m \cdot T_{\ell-1}^{m-1} \tag{18}$$

for all $m = 0, \ldots, k$ and $\ell = j - k + m, \ldots, j$ can be expressed as follows.

Theorem 2 (Brezinski and Walz, 1991). *Let* (16) *be the reference functional* (*with respect to* $(L_{j-k}, \ldots, L_j)$) *of the transformation* (15) *having a characteristic space* $E_k = \operatorname{span}\{f_0, \ldots, f_k\}$. *If*

$$0 \neq V\left| {f_0, \ldots\ldots, f_m \atop L_{\ell-m}, \ldots, L_\ell} \right| := \det(L_\nu f_\mu)_{\nu=\ell-m,\ldots,\ell}^{\mu=0,\ldots,m} \tag{19}$$

for all $m = 0, \ldots, k$ *and* $\ell = j - k + m, \ldots, j$ *and, if* $k \geqslant 2$,

$$0 \neq V\left| {f_1, \ldots\ldots, f_{m-1} \atop L_{\ell-m+1}, \ldots, L_{\ell-1}} \right| \tag{20}$$

for all $m = 2, \ldots, k$ *and* $\ell = j - k + m, \ldots, j$, *then* (15) *results from a triangular recurrence* (18) *where*

$$\lambda_\ell^m = \frac{\omega_\ell^m \cdot R_\ell^{m-1} f_m}{\omega_{\ell-1}^{m-1} \cdot R_\ell^{m-1} f_m - \omega_\ell^{m-1} \cdot R_{\ell-1}^{m-1} f_m}$$

and

$$\mu_\ell^m = \frac{-\omega_\ell^m \cdot R_{\ell-1}^{m-1} f_m}{\omega_{\ell-1}^{m-1} \cdot R_\ell^{m-1} f_m - \omega_\ell^{m-1} \cdot R_{\ell-1}^{m-1} f_m}.$$

for all $m = 1, \ldots, k$ *and* $\ell = j - k + m, \ldots, j$. *In particular, when* $\omega_\ell^m = 1$ *for all* m, ℓ *then*

$$\lambda_\ell^m = \frac{R_\ell^{m-1} f_m}{R_\ell^{m-1} f_m - R_{\ell-1}^{m-1} f_m}, \qquad \mu_\ell^m = 1 - \lambda_\ell^m. \qquad \square \tag{21}$$

5. De Casteljau's algorithm and its characteristic space

In this section we prove that Theorem 2 is applicable to De Casteljau's algorithm, i.e. we present a reference functional and a characteristic space of (9) such that (18) is (10) thus completing some considerations of (Brezinski and Walz, 1991) devoted to this example.

Assuming that $(T_\nu^0)_{\nu\in\mathbb{Z}}$ is a sequence of given control points, we rewrite (9) as a sequence transformation (15) by

$$\beta_{j,\nu}^k = B_{\nu-j+k}^k(t) \quad \text{if } \nu = j-k,\dots,j \tag{22}$$

and $\beta_{j,\nu}^k = 0$ if not where $k = 1,2,3,\dots$ and $j,\nu \in \mathbb{Z}$.

Theorem 3. *Let $F := \mathbb{K}^{\mathbb{Z}}$ be the space of all bi-infinite sequences in $\mathbb{K}$ and let $L_\nu \in F^*$ be defined by*

$$L_\nu f := f(\nu) \quad \text{for all } f \in \mathbb{K}^{\mathbb{Z}},\ \nu \in \mathbb{Z}.$$

Let $f_j \in F$ be defined by $f_0(\nu) = 1$ and $f_j(\nu) = \nu^{j-1}\cdot(1-1/t)^\nu$ for $j \geqslant 1$, $\nu \in \mathbb{Z}$, where $0 < t < 1$ is a parameter.

Then, (16) is a reference functional of (15) (with respect to $(L_{j-k},\dots,L_j)$) with a characteristic space $E_k = \operatorname{span}\{f_0,\dots,f_k\}$. From Theorem 2 we obtain $\lambda_\ell^m = 1-t$, i.e. (18) is De Casteljau's algorithm (10).

Proof. Assume that F and (L_ν) are defined as in the theorem. Then, (16) is a reference functional for (15) which for $k = j = n$ is (9) (due to (22)). In order to motivate the definitions of $f_1,\dots,f_k$ given in the theorem, we derive them from the condition

$$R_\nu^k f_j = 0 \quad \text{for all } \nu \in \mathbb{Z}.$$

Explicitly, this is a homogeneous linear difference equation of order k

$$B_0^k(t)\cdot f_j(\nu-k) + \dots + B_k^k(t)\cdot f_j(\nu) = 0.$$

Its characteristic polynomial (written in λ) is

$$\sum_{\ell=0}^k \lambda^\ell \cdot B_\ell^k(t) = (1-t+\lambda\cdot t)^k$$

having the root $\lambda := 1-1/t$ of multiplicity k. Taking the k linearly independent solutions $f_1,\dots,f_k$ leads to the definitions given in the theorem and assumed in the sequel.

The vectors $f_0, f_1,\dots,f_k$ and the functionals (L_ν) are linearly independent and define the characteristic functional (16) (with coefficients from (22)) of (9) as well as a characteristic space $E_k = \operatorname{span}\{f_0,\dots,f_k\}$. Hence, it remains to show that the assumptions of Theorem 2 hold.

Note that by exploiting $R_{m+j}^j f_i = \delta_{i,0}\cdot\omega_{m+j}^i$ for $i = 0,\dots,j$ we have recursively

$$V\left|{}^{f_0,\dots,f_m}_{L_{j-m},\dots,L_j}\right| = \frac{1}{\beta_{j,j}^m} V\left|{}^{f_0,\dots\dots,f_{m-1},f_m}_{L_{j-m},\dots,L_{j-1},R_j^m}\right| = \frac{(-1)^j}{\beta_{j,j}^m} V\left|{}^{f_0,\dots\dots,f_{m-1}}_{L_{j-m},\dots,L_{j-1}}\right| \neq 0$$

starting with $V|{}^{f_0}_{L_{j-m}}| = 1$. This proves (19).

Dividing each row by a suitable power of λ in $V|{}^{f_1,\dots\dots,f_{m-1}}_{L_{j-m+1},\dots,L_{j-1}}|$ one gets a (classical) Vandermonde determinant. From this one concludes that also (20) holds.

Finally, some direct calculations (starting from (16) and (22) and the definitions given in the theorem) prove $\omega_\ell^m = 1$ and

$$\begin{aligned} R_\ell^{m-1} f_m &= (t-1)^\ell t^{-\ell+m-1} \cdot \sum_{\mu=0}^{m-1} \binom{m-1}{\mu} (-1)^{m-1-\mu} (\ell - m + 1 + \mu)^{m-1} \\ &= (m-1)!(t-1)^\ell t^{-\ell+m-1} \end{aligned}$$

where we observed that the sum is a forward difference of order $m-1$ with step size 1 of $(\ell - m + 1 + \cdot)^{m-1}$ and hence equal to $(m-1)!$. Using this in (21) and $\omega_\ell^m = 1$ we find $\lambda_\ell^m = 1 - t$. □

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