This article presents some numerical examples for coupling the finite element method (FEM) and the boundary element method (BEM) as analyzed in [1]. This coupling procedure combines the advantages of boundary elements (problems in unbounded regions) and of finite elements (nonlinear problems with inhomogeneous data). In [28], experimental rates of convergence for the \( h \) version are presented, where the accuracy of the Galerkin approximation is achieved by refining the mesh. In this article we treat the \( h-p \) version, combining an increase of the degree of the piecewise polynomials with a certain mesh refinement. In our model examples, we obtain theoretically and numerically exponential convergence, which indicates a great efficiency in particular if singularities appear.

1. INTRODUCTION

The finite element method can be applied to nonlinear or inhomogeneous problems concerning partial differential equations, but is restricted to bounded domains. This is contrary to the boundary element methods, which can be applied to the most important linear and homogeneous partial differential equations with constant coefficients also in unbounded domains (provided that the boundary is bounded).

The coupling of FEM and BEM comes of interest, since it allows a combination of the advantages of both methods. Hence, it is applied for linear transmission problems in scattering problems, elastodynamics, electromagnetics, and elasticity [1–6]; numerical examples may be found in [7, 8]. Recently, a class of nonlinear interface problems is treated in [9–12] using a symmetric coupling method, which allows a variational formulation of a saddle-point problem.

In this article we improve the convergence of the coupling method using the \( h-p \) version with a geometric mesh for the first time. Even in the case of singular solutions, we get exponential convergence, which leads to an efficient numerical treatment of the problems.

A motivating interface problem in three-dimensional solid mechanics and a two-dimensional numerical test case are stated in Sections II and III to recall the coupling
procedure and to describe the error analysis. In particular we contribute an estimate for approximate discrete solutions in Theorem 3. For numerical results in two-dimensional elasticity, we refer to [16]; this article focuses on two-dimensional harmonic examples. In Section IV, the discretization for the finite elements and boundary elements is sketched for the h–p version. Then, we derive exponential convergence of the h–p version of the Galerkin procedure of the coupled problem. The iterative solution and its numerical implementation are described explicitly in Sections V and VI. Numerical experiments are reported in Section VII to underline the exponential convergence and the efficiency of the proposed treatment of such nonlinear interface problems in case of singularities.

II. COUPLING METHOD FOR A MONOTONE PROBLEM FOR HENCKY-ELASTICITY

Let \( \Omega_1 \) be a three-dimensional bounded Lipschitz domain with \( \partial \Omega_1 = \Gamma_u \cup \Gamma \) in which we assume the nonlinear Hencky–von Mises stress–strain relation of the form

\[
\sigma = \left( k - \frac{2}{3} \mu(\gamma) \right) I \cdot \text{div} \ u_1 + 2 \mu(\gamma) \varepsilon,
\]

where \( \sigma \) and \( \varepsilon = \frac{1}{2} (\nabla u^T + \nabla u) \) denotes the (Cauchy) stresses and the (linear Green) strain, respectively, see [13–15]. Then, if we define

\[
P_1(u_1)_i := \frac{\partial}{\partial x_i} \left( k - \frac{2}{3} \mu(\gamma(u_1)) \right) \cdot \text{div} \ u_1 + \sum_{j=1}^{3} 2 \frac{\partial}{\partial x_j} \mu(\gamma(u_1)) \varepsilon_{ij}(u_1)
\]

for \( i = 1, 2, 3 \), the equilibrium condition \( \text{div} \ \sigma + F = 0 \) gives

\[
P_1(u_1) = F \quad \text{in} \ \Omega_1.
\]

Here, the bulk modulus \( k \) and the function \( \mu(\gamma) \) in \( P_1 \) satisfy (cf., e.g., [14])

\[
0 < \mu_0 \leq \mu(\gamma) \leq \frac{3}{2} k, \quad 0 < \mu_1 \leq \mu + 2 \gamma \frac{d \mu}{d \gamma} \leq \mu_2 < \infty,
\]

where \( \mu_0, \mu_1, \mu_2 \) are constants and

\[
\gamma(u_1) = \sum_{i,j=1}^{3} \left( \varepsilon_{ij} - \delta_{ij} \frac{1}{3} \cdot \text{div} \ u_1 \right)^2, \quad \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_{1i}}{\partial x_j} + \frac{\partial u_{1j}}{\partial x_i} \right).
\]

In a surrounding unbounded exterior region \( \Omega_2 \), we consider the homogeneous Lamé system describing linear isotropic elastic material, with the Lamé constants \( \mu_2 > 0 \), \( 3 \lambda_2 + 2 \mu_2 > 0 \),

\[
P_2(u_2) = -\mu_2 \Delta u_2 - (\lambda_2 + \mu_2) \text{grad} \ \text{div} \ u_2 = 0 \quad \text{in} \ \Omega_2.
\]

The interface problem under consideration [11] reads: For a given vector field \( F \) in \( \Omega_1 \) find vector fields \( u_j \) in \( \Omega_j (j = 1, 2) \) satisfying \( u_1|_{\Gamma_u} = 0 \), the differential Eqs. (1), (2), the interface conditions

\[
u_1 = u_2, \quad T_1(u_1) = T_2(u_2) \quad \text{on} \ \Gamma,
\]

(3)
and the regularity condition at infinity \((n = 3)\)

\[
u_2 = \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty.
\]  

(4)

Here, with \(\mu_1 = \mu(\gamma(u_1)), \lambda_1 = k - \frac{2}{3} \mu(\gamma(u_1))\), the tractions are given by

\[
T_j(u_j) = 2 \mu_j \partial_n u_j + \lambda_j n \div u_j + \mu_j n \times \curl u_j,
\]  

(5)

and \(\partial_n u_j\) is the derivative with respect to the outer normal on \(\Gamma\).

We are interested in solutions \(u_j\) of \((1)-(4)\), which belong to \((H^{1,\infty}(\Omega_j))^{3}\), i.e., which are of finite energy. A variational formulation is obtained as in [11]. An application of the first Green formula to \((1)\) yields

\[
\int_{\Omega_1} P_1 u_1 w \, dx = \Phi_1(u_1, w) - \int_{\Gamma} T_1 u_1 w \, ds
\]  

(6)

for all \(w \in H^1(\Omega_1)\), where

\[
\Phi_1(u_1, w) := \int_{\Omega_1} \left\{ k - \frac{2}{3} \mu(\gamma(u_1)) \div u_1 \div w + \sum_{i,j=1}^3 2 \mu(\gamma(u_1)) \epsilon_{ij}(u_1) \epsilon_{ij}(w) \right\} dx.
\]  

(7)

On the other hand, the solution \(u_2\) of \((2)\) is given by the Somigliana representation formula for \(x \in \Omega_2:\)

\[
u_2(x) = \int_{\Gamma} \{ T_2(x, y)v_2(y) - G_2(x, y)\phi_2(y) \} \, ds(y),
\]  

(8)

where \(v_2 = u_2, \phi_2 = T_2(u_2)\) on \(\Gamma\), and the fundamental solution \(G_2(x, y)\) of \(P_2 u_2 = 0\) is the \(3 \times 3\) matrix function

\[
G_2(x, y) = \frac{\lambda_2 + 3 \mu_2}{8 \pi \mu_2 (\lambda_2 + 2 \mu_2)} \left\{ \frac{1}{|x - y|} I + \frac{\lambda_2 + \mu_2}{\lambda_2 + 3 \mu_2} \frac{(x - y)(x - y)^T}{|x - y|^3} \right\}
\]  

with the unit matrix \(I\) and \(T_2(x, y) = T_{2,y}(G_2(x, y))^T\), where \(T\) denotes transposition. Taking Cauchy data in \((8)\), i.e., boundary values and tractions on \(\Gamma\) for \(x \to \Gamma\), we obtain a system of boundary integral equations on \(\Gamma,\)

\[
v_2 = \left( \frac{1}{2} + \Lambda_2 \right) v_2 - V_2 \phi_2 \quad \text{and} \quad \phi_2 = -W_2 v_2 + \left( \frac{1}{2} - \Lambda_2' \right) \phi_2
\]  

(9)

with the single layer potential \(V_2\), a weakly singular boundary integral operator, the double layer potential \(\Lambda_2\) and its dual \(\Lambda_2'\), strongly singular operators, and the hypersingular operator \(W_2\) defined as

\[
V_2 \phi_2(x) = \int_{\Gamma} G_2(x, y) \phi_2(y) \, ds(y)
\]  

\[
\Lambda_2 v_2(x) = \int_{\Gamma} T_2(x, y)v_2(y) \, ds(y)
\]  

\[
\Lambda_2' \phi_2(x) = T_{2,x} \int_{\Gamma} G_2(x, y)^T \phi_2(y) \, ds(y)
\]  

\[
W_2 v_2(x) = -T_{2,x} \int_{\Gamma} T_2(x, y)v_2(y) \, ds(y).
\]
As in interface problems for purely linear equations [3], we obtain a variational formulation for the interface problem (1)–(4) by adding a weak form of the boundary integral Eqs. (9) on \( \Gamma \) to the weak form (6). Then we insert it into (6) and make use of the interface conditions (3), i.e., \( t_2 = t_1 =: \phi \) and \( v_2 = u_1 =: u \).

This yields the following variational problem: For given \( F \in L^2(\Omega_1)^3 \) find \( u \in H^1(\Omega_1)^3 \), \( \phi \in H^{-1/2}(\Gamma)^3 \) such that \( u \big|_{\Gamma_u} = 0 \) and

\[
b(u, \phi; w, \psi) = \int_{\Omega_1} F \cdot w \, dx \quad \text{for all} \quad (w, \psi) \in H^1(\Omega_1)^3 \times H^{-1/2}(\Gamma)^3.
\]  

(10)

Here, with the form \( \Phi_1(\cdot, \cdot) \) in (7) and the brackets \( \langle \cdot, \cdot \rangle \) denoting the extended \( L^2 \)-duality duality between the trace space \( H^{1/2}(\Gamma)^3 \) and its dual \( H^{-1/2}(\Gamma)^3 \), we define

\[
b(u, \phi; w, \psi) := \Phi_1(u, w) + \langle w, W_2 u \rangle - \left\langle w, \left( \frac{1}{2} - \Lambda_2' \right) \phi \right\rangle - \left\langle \left( \frac{1}{2} - \Lambda_2 \right) u, \psi \right\rangle
\]

\[
- \langle \psi, V_2 \phi \rangle.
\]  

(11)

**Theorem 1 ([11, 16]).** For \( F \in L^2(\Omega_1)^3 \) there exists exactly one solution \( u \in H^1(\Omega_1)^3 \), \( \phi \in H^{-1/2}(\Gamma)^3 \) of (10) yielding \( u = u_1 \) in \( \Omega_1 \) and \( u_2 \) given by (8) in \( \Omega_2 \) a solution of the interface problem (1)–(4).

The proof in [11] is based on the fact that the \( C^2 \)-functional,

\[
J_1(u, \phi) := A(u) + \frac{1}{2} \langle u, W_2 u \rangle
\]

\[
- \int_{\Omega_1} F u \, dx + \left\langle \phi, \left( \Lambda_2 - \frac{1}{2} \right) u \right\rangle - \frac{1}{2} \langle \phi, V_2 \phi \rangle
\]

\[
A(u) := \int_{\Omega_1} \left\{ \frac{1}{2} |\text{div} \, u|^2 + \int_0^{\gamma(t)} \mu(t) \, dt \right\} \, dx,
\]  

(12)

\( u \in H^1(\Omega_1)^3 \), \( \phi \in H^{-1/2}(\Gamma)^3 \), has a unique saddle-point. The two-dimensional case, treated in [16], requires minor modifications only.

Given finite dimensional subspaces \( X_N \times Y_M \) of \( H^1(\Omega_1)^3 \times H^{-1/2}(\Gamma)^3 \), the Galerkin solution \( (u_N, \phi_M) \) in \( X_N \times Y_M \) is the unique saddle-point of the functional \( J_1 \) on \( X_N \times Y_M \); the Galerkin scheme for (10) reads: Given \( F \in L^2(\Omega_1)^3 \) find \( u_N \in X_N \) and \( \phi_M \in Y_M \) such that, for all \( w \in X_N \) and \( \psi \in Y_M \),

\[
b(u_N, \phi_M; w, \psi) = \int_{\Omega_1} f \cdot w \, dx.
\]  

(13)

The Theorem 2 states quasi-optimal convergence in the energy norm for any conforming Galerkin scheme. See [16] for the two-dimensional case.

**Theorem 2 ([11, 16]).** There exists exactly one solution \( (u_N, \phi_M) \in X_N \times Y_M \) of the Galerkin Eqs. (13). There exists a constant \( C \) independent of \( X_N \) and \( Y_M \) such that

\[
\| u - u_N \|_{H^1(\Omega_1)^3} + \| \phi - \phi_M \|_{H^{-1/2}(\Gamma)^3}
\]

\[
\leq C \left\{ \inf_{w \in X_N} \| u - w \|_{H^1(\Omega_1)^3} + \inf_{\psi \in Y_M} \| \phi - \psi \|_{H^{-1/2}(\Gamma)^3} \right\}
\]  

(14)
where \((u, \phi) \in H^1(\Omega_1)^3 \times H^{-1/2}(\Gamma)^3\) is the exact solution of the variational problem (10).

Within the class of saddle-point problems, the Galerkin solution can, in general, be approximated by an iterative process only. To control the error of an approximation \((\tilde{u}_N, \tilde{\phi}_M)\) to the Galerkin solution \((u_N, \phi_M)\), we prove the following a posteriori estimate.

**Theorem 3.** Let \((u_N, \phi_M) \in X_N \times Y_M\) be the unique Galerkin solution of (13) and let \((\tilde{u}_N, \tilde{\phi}_M) \in X_N \times Y_M\) be known such that we can compute

\[
\tilde{r}_N := \|DJ_1 \times (\tilde{u}_N, \tilde{\phi}_M)\|_{H^1(\Omega_1)^3 \times H^{-1/2}(\Gamma)}.
\]

Then,

\[
\|(u_N - \tilde{u}_N, \phi_M - \tilde{\phi}_M)\|_{H^1(\Omega_1)^3 \times H^{-1/2}(\Gamma)} \leq C \cdot \tilde{r}_N.
\]

The constant \(C > 0\) depends on \(\Omega_1, \Gamma,\) and the constants \(k, \mu, \lambda_2, \mu_2\) only; but not on \(X_N \times Y_M\).

**Proof.** Since \(V_2\) is positive definite and \(W_2\) is positive semi-definite, and since \(D^2A\) is uniformly monotone (see, e.g., [11]) we infer, using the main theorem on calculus,

\[
C^{-1}\|(u_N - \tilde{u}_N, \phi_M - \tilde{\phi}_M)\|_{H^1(\Omega_1)^3 \times H^{-1/2}(\Gamma)} \leq 2 \int_0^1 D^2A(t \cdot u_N + (1 - t) \cdot \tilde{u}_N)\times[u_N - \tilde{u}_N, u_N - \tilde{u}_N]dt
\]

\[+
\langle V_2(\phi_M - \tilde{\phi}_M), (\phi_M - \tilde{\phi}_M)\rangle
\]

\[+
\langle W_2(u_N - \tilde{u}_N), (u_N - \tilde{u}_N)\rangle
\]

\[\leq 2DA(u_N)[u_N - \tilde{u}_N] - DA(\tilde{u}_N)[u_N - \tilde{u}_N]
\]

\[+
\langle V_2(\phi_M - \tilde{\phi}_M), (\phi_M - \tilde{\phi}_M)\rangle
\]

\[+
\langle W_2(u_N - \tilde{u}_N), (u_N - \tilde{u}_N)\rangle.
\]

Noting that \(DJ_1(u_N)[u_N - \tilde{u}_N, \tilde{\phi}_M - \phi_M] = 0\) and \(DA = \Phi_1\), we derive

\[
C^{-1}\|(u_N - \tilde{u}_N, \phi_M - \tilde{\phi}_M)\|_{H^1(\Omega_1)^3 \times H^{-1/2}(\Gamma)} \leq -DJ_1(\tilde{u}_N, \tilde{\phi}_N)[(u_N - \tilde{u}_N, \tilde{\phi}_M - \phi_M)]
\]

\[\leq \tilde{r}_N \cdot \|(u_N - \tilde{u}_N, \tilde{\phi}_M - \phi_M)\|_{H^1(\Omega_1)^3 \times H^{-1/2}(\Gamma)}.
\]

From this, we conclude the assertion.

In the numerical examples below, we compute \((\tilde{u}_N, \tilde{\phi}_M)\) such that \(\tilde{r}_N\) is of machine precision. Then, by triangle inequality, Theorems 2 and 3 verify that \((\tilde{u}_N, \tilde{\phi}_M)\) is a reasonable approximation of \((u, \phi)\). This justifies the numerical treatment below and in [8].

### III. MODEL PROBLEM

Our numerical experiments with the h–p version are related to the following two-dimensional model problem [8] involving prescribed jumps across the interface \(\Gamma\): **Given** \(F \in L^2(\Omega), f \in H^{1/2}(\Gamma), g \in H^{-1/2}(\Gamma), \text{find } u_1 \in H^1(\Omega_1), u_2 \in H^1_{loc}(\Omega_2)\) satisfying
Here, $A \in \mathbb{R}$ is a constant depending on $u_2$ and $p \in C^1(\mathbb{R})$ satisfies, with some constants $\gamma_1, \gamma_2 > 0$,

$$\gamma_1 \leq p(r) \leq \gamma_2 \text{ and } \gamma_1 \leq p(r) + rp'(r) \leq \gamma_2 \quad (r \geq 0).$$

As in the previous section, the interface problem (15) allows an equivalent variational formulation:

$$b(u, \phi; w, \psi) = \int_{\Omega_1} Fw \, dx + l(w, \psi)$$

for all $(w, \psi) \in H^1(\Omega_1) \times H^{-1/2}(\Gamma)$, where $b$ is given in (11), and

$$\Phi(u, w) = 2 \cdot \int_{\Omega_1} (p|\nabla u|\nabla w + uw) \, dx$$

$$l(w, \psi) = \langle w, g \rangle + \langle f, \psi \rangle + \langle w, W_2f \rangle + \langle \Lambda_2'g, w \rangle - \langle \Lambda_2f, \psi \rangle + \langle V_2g, \psi \rangle. \quad (18)$$

Corresponding to the Laplace operator, we have the single-layer potential operator $V_2$, the double-layer potential operator $\Lambda_2$ and its adjoint $\Lambda_2'$, and the hypersingular operator $W_2$ as defined above with $-\frac{1}{\pi} \log|x-y|$ replacing $G_2(x,y)$ and $\frac{\partial}{\partial n}$ replacing $T$ (see, e.g., [8] for details). As in [8], we assume $\text{cap}(\Gamma) < 1$ so that $V_2$ is positive definite. Let

$$J_1(u, \phi) = J(u) + \langle u, (\Lambda_2' - 1)\phi \rangle - \frac{1}{2} \langle \phi, V_2\phi \rangle$$

$$J(u) = 2J_0(u) + \frac{1}{2} \langle u, W_2u \rangle$$

$$J_0(u) := \int_{\Omega_1} \left\{ \int_0^{\|u\|} tp(t) \, dt + \frac{1}{2} |u|^2 - fu \right\} \, dx.$$

Under the present conditions on $p$, the second Gateaux derivative of $J_0$ is uniformly monotone [8], so that the results in [11] are applicable and briefly summarized as follows (see [8, 16]):

a. The weak form of the Euler equation to the variational problem of $J_1$ coincides with the weak form (16) of the coupling problem (15).

b. The variational problem (16) has exactly one solution $(u, \phi)$.

c. For any pair of finite dimensional subspaces $X_N \subset H^1(\Omega), Y_M \subset H^{-1/2}(\Gamma)$, there exists exactly one solution $(u_N, \phi_M)$ of the Galerkin scheme for (16) and a constant $C$ independent of $X_N$ and $Y_M$ such that

$$\|u - u_N\|_{H^1(\Omega)} + \|\phi - \phi_M\|_{H^{-1/2}(\Gamma)} \leq C \left\{ \inf_{w \in X_N} \|u - w\|_{H^1(\Omega)} + \inf_{\phi \in Y_M} \|\phi - \psi\|_{H^{-1/2}(\Gamma)} \right\}.$$

d. Theorem 3 is also valid for the two-dimensional model problem at hand.
IV. DISCRETIZATION

Let the two-dimensional domain $\Omega_1$ have the polygonal boundary $\Gamma$, i.e., $\Gamma = \bigcup_{j=1}^{m} \Gamma_j$ is the union of straight lines $\Gamma_1, \ldots, \Gamma_m$ connecting the endpoints $x_0 = x_m, x_1, \ldots, x_m$. Near the corner point $x_j$ we improve the approximation quality of the trial space concerning the corner singularities using a geometric mesh and a particular distribution of the polynomial degrees.

First we define a geometric partition $I^n_\sigma$ of level $n$ on the interval $I = [0, 1]$ by $x_0 := 0$ and $x_j := \sigma^{n-j}, j = 1, \ldots, n$. With a degree vector $q = (q_1, \ldots, q_n)$ the trial space $S^q(I^n_\sigma)$ is the vector space of all continuous functions on $I$, which are piecewise polynomials with degree $q_{j+1}$ on $(x_j, x_{j+1})$. Next we introduce the analogous two-dimensional vector space on $Q = [0, 1] \times [0, 1]$ as a space of tensor-products

$$S^{q,r}(Q^n_\sigma) = S^q(I^n_\sigma) \times S^r(I^n_\sigma).$$

In our examples, we use a geometric mesh-refinement towards the origin of $Q$ by using a geometric partition of $\Omega_1$ obtained by affine transformations of $S^{q,r}(Q^n_\sigma)$ as shown in Fig. 1 and Fig. 2. Then we define $X_N := S^{q,r}(Q^n_\sigma)$ with $N$ being the dimension of $S^{q,r}(Q^n_\sigma)$.

The trial space $Y_M$ for the boundary elements is obtained as a trace space of gradients in $X_N$, i.e.,

$$Y_M := S^r(\Gamma^n_\sigma) := \{\nabla w_N \mid \Gamma; w_N \in X_N\},$$

where $M := \text{dim } Y_M$ is the number of degrees of freedom. This means we take the partition of the boundary $\Gamma$ induced by the geometric partition of $\Omega_1$ and take piecewise polynomials there with the degree from the neighboring finite element (along the current side) minus one. Note functions in $Y_M$ are, in general, discontinuous.

By using countable normed spaces $B^l_\beta(\Omega)$ (which are appropriately weighted Sobolev spaces; see Appendix) used by Guo and Babuska in [17], one can prove convergence rates (see [8]) as in the linear case [7]: Denote the internal angle at $x_j$ by $\omega_j (0 < \omega_j < 2\pi, 1 \leq j \leq m)$ and choose $\beta = (\beta_1, \ldots, \beta_m)$ under the condition $0 < \beta_j < 1/2, \beta_j > 1 - \pi/\omega_j$. In the linear case, certain conditions on the data $f$ and $g$ (namely $f \in B^{3/2}_\beta(\Gamma)$ and $g \in B^{1/2}_\beta(\Gamma)$) lead to the regularity of the solution (namely $u \in B^l_\beta(\Omega)$). In the nonlinear case, we have to assume this regularity assumption explicitly and then conclude, as in [7],

![Fig. 1. Geometric mesh with polynomial degrees.](image-url)
V. SOLVING THE DISCRETE PROBLEM

According to the nonlinear function $J_1$ as in (19), the Galerkin equations

\[ DJ_1(u_N, \phi_M) = 0 \quad \forall \ (v, \phi) \in X_N \times Y_N, \]

are to be solved within an iterative process. Let $U_N^{(m)}$ and $\Phi_M^{(m)}$ denote the coefficient vectors of the piecewise polynomials $u_N^{(m)}$ and $\phi_M^{(m)}$, respectively, obtained iteratively with Newton–Raphson method or the method of Broyden. One step of Newton’s–Raphson’s method can be written in a compact form as

\[
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\begin{pmatrix}
U_N^{(m)} - U_N^{(m+1)} \\
\Phi_M^{(m)} - \Phi_M^{(m+1)}
\end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},
\]

with $A_{11}$ being positive definite and $A_{22}$ being negative definite defined by

\[
A_{11} := D^2J_0(u_N^{(m)}) [v; w] + \langle w, W_2 v \rangle,
\]

\[
A_{12} := \langle (\Lambda_2' - 1) \xi, v \rangle = A_{21}^T,
\]

\[
A_{22} := - \langle V_2 \xi, \psi \rangle
\]

\[
B_1 := DJ_0(u_N^{(m)}) [v] + \langle u_N^{(m)} - f, W_2 v \rangle + \langle (\Lambda_2' - 1) \phi_M^{(m)}, v \rangle - \langle (\Lambda_2 + 1) g, v \rangle,
\]

\[
B_2 := - \langle V_2 \phi_M^{(m)} - g - (\Lambda_2 - 1) u_N^{(m)} + (\Lambda_2 - 1) f, \psi \rangle.
\]

One iteration step of the method of Broyden, a quasi-Newton method, reads

\[
\begin{pmatrix}
U_N^{(m+1)} \\
\Phi_M^{(m+1)}
\end{pmatrix} = \begin{pmatrix}
U_N^{(m)} \\
\Phi_M^{(m)}
\end{pmatrix} + A_0^{-1} \cdot R_m,
\]

where $A_0 = (A_{ij})$ is the stiffness matrix evaluated at $u_N^{(0)}, \phi_M^{(0)}$ and then updated by

\[
A_m = A_{m-1} + \frac{1}{d_m} \cdot \frac{d_m}{d_m} (e_m - A_{m-1} d_m) d_m^T,
\]

where $d_m = A_m^{-1} \cdot A_0^{-1} \cdot R_m.$
while \( d_m := U_N^{(m)} - U_N^{(m-1)}, e_m := R_m - R_{m-1}, R_{-1} := U_N^{(-1)} := 0 \).

In our numerical examples, the iterations of the Newton– and Broyden–method have been performed until the residual \( \tilde{r}_N \) in Theorem 3 was of the order of the machine precision \( \epsilon \). Then, Theorem 3 verifies that the computed approximation \( (u_N^{(m)}, \phi_N^{(m)}) \) might replace the unknown Galerkin–solution \( (u_N, \phi_N) \) in our numerical experiments reported below.

### VI. NUMERICAL IMPLEMENTATION

In this subsection, we briefly report on the numerical evaluation of the stiffness matrices involved in the iterative process of Section V.

#### A. Integrals over the Domain

In the evaluation of the Gateaux–derivatives \( DJ_0 \) and \( D^2J_0 \) of \( J_0 \) [see (20)] we have integrals over \( \Omega_1 \) to be computed by applying a standard \( 32 \times 32 \) point Gaussian quadrature formula on any element.

#### B. Single-Layer Potential

With \( \phi \) and \( \psi \in S'(\Gamma_\alpha) \) we get for the single-layer potential operator \( V_2 \):

\[
\langle V_2 \phi, \psi \rangle = -\frac{1}{\pi} \int_\Gamma \psi(y) \int_\Gamma \phi(x) \log|x - y| \, ds_x \, ds_y,
\]

where \( \psi \) and \( \phi \) are monomials on \( \Gamma_j \in \Gamma_\alpha \). To perform the outer integral, we use a 32-point Gauss quadrature formula, whereas the inner integral we compute analytically as follows: An affine transformation mapping \( \Gamma_j \) to \([-1, 1]\] leads to

\[
I_1(y) := \int_{\Gamma_j} \phi(x) \log|x - y| \, ds_x = \frac{ds_x}{2d\xi} \int_{-1}^{1} \xi^r \log(a \xi^2 + b \xi + c) \, d\xi
\]

\[
= \frac{ds_x}{2d\xi} \int_{-1}^{1} \xi^r \log(|a|) \, d\xi + \Re \frac{ds_x}{2d\xi} \int_{-1}^{1} \xi^r \log(\xi - z_1) \, d\xi
\]

\[
+ \Re \frac{ds_x}{2d\xi} \int_{-1}^{1} \xi^r \log(\xi - z_2) \, d\xi,
\]

where the constants \( a, b, c \) with \( b^2 - 4ac \leq 0 \), depend on \( y \) only, and \( \Gamma_j \) and \( z_1 \) and \( z_2 \) are complex numbers with \((\xi - z_1)(\xi - z_2) = \xi^2 + \frac{b}{a} \xi + \frac{c}{a}\). The appearing integrals are then evaluated with

\[
\int \xi^r \log(\xi - z_0) \, d\xi = \frac{\xi^{1+r} - z_0^{1+r}}{1 + r} \log(\xi - z_0) - \frac{1}{1 + r} \sum_{k=1}^{1+r} \frac{x^{r-k+2}z_0^{k-1}}{r - k + 2}.
\]
C. Double-Layer Potential

For \( \nu \in S^{p-q}(\Omega_\nu) \) and \( \psi \in S'(\Gamma^\nu) \) a typical term involving the double-layer potential operator is

\[
\langle \psi, \Lambda_2 \nu \rangle = \langle \Lambda_2 \psi, \nu \rangle = -\frac{1}{\pi} \int_\Gamma \nu(y) \int_\Gamma \psi(x) \frac{\partial}{\partial n_y} \log|x-y| \, ds_x \, ds_y,
\]

where \( \psi \) is a monomial on \( \Gamma_j \subset \Gamma^\nu \). The outer integral we evaluate using a 32-point Gauss quadrature formula, whereas the inner integral we again compute analytically: With an affine transformation and related constants \( a, b, c, e, f \) satisfying \( b^2 - 4ac \leq 0 \),

\[
I_2(y) := \int_{\Gamma_j} \phi(x) \frac{\partial}{\partial n_x} \log|x-y| \, ds = \int_{-1}^{1} \xi' e^{\xi} + f \, d\xi.
\]

To evaluate the appearing integrals we let \( R := a\xi^2 + b\xi + c \), \( \Delta := 4ac - b^2 \), and make use of

\[
\int \frac{\xi^m}{R} d\xi = \frac{\xi^{m-1}}{(m-1)A} - \frac{B}{A} \int \frac{\xi^{m-1}}{R} d\xi - \frac{C}{A} \int \frac{\xi^{m-2}}{R} d\xi
\]

\[
\int \frac{\xi}{R} d\xi = \frac{1}{2A} \log(R) - \frac{B}{2A} \int \frac{d\xi}{R}
\]

\[
\int \frac{d\xi}{R} = \frac{2}{\sqrt{\Delta}} \arctan\left(\frac{2\Lambda\xi + B}{\sqrt{\Delta}}\right).
\]

D. Hypersingular Operator

For \( u, \nu \in S^{p-q}(\Omega_\nu) \), we evaluate the hypersingular operator \( W_2 \) with procedures of the single-layer potential operator (see [18]): \( \langle u, W_2 \nu \rangle = -\langle \nu, W_2 u \rangle \).

VII. NUMERICAL RESULTS

For the computations we consider a couple of examples for the interface problem with \( \Omega_1 \) being the square \( \{(x_1, x_2) \in \mathbb{R}^2: |x_i| < 1, i = 1, 2\} \). In all examples we have

\[
p(r) = 2 + \frac{1}{1+r} \quad (r \geq 0),
\]

so that \( 1 \leq p(r) \leq 3, 1 \leq p(r) + r \cdot p'(r) \leq 3, r > 0 \). With \( G(r) = \int_0^1 tp(t) \, dt = r^2 + r - \log(1 + r) \), the functional \( J_0 \) on \( H^1(\Omega_1) \) becomes

\[
J_0 = \int_{\Omega_1} \left\{ |\nabla u|^2 + |\nabla u| - \log(1 + |\nabla u|) + \frac{1}{2} |u|^2 - 2F \cdot u \right\} \, dx,
\]

and with (15) we have \( P_1 u = -2\Delta u - div(\frac{\nabla u}{1+|\nabla u|}) + u \). In Tables I–III and Figs. 3, 4, and 5, we present experimental rates of convergence for the \( L_2 \)-errors \( e := \|u - u^{(m)}_N\|_{L_2(\Omega_1)} \) in \( \Omega_1 \) and \( \epsilon := || \phi - \phi^{(m)}_N ||_{L_2(\Gamma)} \) on \( \Gamma \), where \( u_1 \in H^1(\Omega_1) \) and \( \phi = p((\nabla u_1)^{N_1})^N_{\partial \Gamma} \in H^{-1/2}(\Gamma) \) solve the interface problem (15). In the sequel, \( m_K \) denotes the number of iterations of the Newton–method, and \( N_1 \) and \( N_2 \) denote the dimensions of \( S^{p,q} \) and \( S' \), respectively.
### TABLE I. Absolute errors in Example 1.

<table>
<thead>
<tr>
<th>$|u - u_N^{(m_K)}|_{L^2(\Omega_1)}$</th>
<th>$N_1$</th>
<th>$|\phi - \phi_M^{(m_K)}|_{L^2(\Gamma_1)}$</th>
<th>$N_2$</th>
<th>$m_K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.11358</td>
<td>4</td>
<td>1.5117</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>0.03013</td>
<td>17</td>
<td>1.3554</td>
<td>14</td>
<td>5</td>
</tr>
<tr>
<td>0.01181</td>
<td>48</td>
<td>1.1910</td>
<td>30</td>
<td>5</td>
</tr>
<tr>
<td>0.00518</td>
<td>112</td>
<td>1.0433</td>
<td>52</td>
<td>5</td>
</tr>
</tbody>
</table>

* $\sigma = 0.5$:

| 0.11358                             | 4     | 1.5117                                      | 4     | 5     |
| 0.01324                             | 17    | 1.1679                                      | 14    | 5     |
| 0.002815                            | 48    | 0.9107                                      | 30    | 5     |
| 0.000989                            | 112   | 0.7141                                      | 52    | 5     |

* $\sigma = 0.25$:

| 0.11358                             | 4     | 1.5117                                      | 4     | 5     |
| 0.01201                             | 17    | 1.02482                                     | 14    | 5     |
| 0.00422                             | 48    | 0.70973                                     | 30    | 5     |

* $\sigma = 0.1$:

| 0.11358                             | 4     | 1.5117                                      | 4     | 5     |
| 0.01201                             | 17    | 1.02482                                     | 14    | 5     |
| 0.00422                             | 48    | 0.70973                                     | 30    | 5     |

### TABLE II. Absolute errors in Example 2.

<table>
<thead>
<tr>
<th>$|u - u_N^{(m_K)}|_{L^2(\Omega_1)}$</th>
<th>$N_1$</th>
<th>$|\phi - \phi_M^{(m_K)}|_{L^2(\Gamma_1)}$</th>
<th>$N_2$</th>
<th>$m_K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.47016</td>
<td>4</td>
<td>3.27856</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>0.07507</td>
<td>24</td>
<td>2.11075</td>
<td>16</td>
<td>5</td>
</tr>
<tr>
<td>0.005175</td>
<td>84</td>
<td>0.7572</td>
<td>36</td>
<td>5</td>
</tr>
<tr>
<td>0.000695</td>
<td>217</td>
<td>0.2585</td>
<td>64</td>
<td>5</td>
</tr>
</tbody>
</table>

* $\sigma = 0.25$:

| 2.47016                              | 4     | 3.27856                                      | 4     | 2     |
| 0.03830                              | 24    | 1.86021                                      | 16    | 5     |
| 0.01073                              | 84    | 0.56459                                      | 36    | 5     |
| 0.001690                             | 217   | 0.170889                                     | 64    | 5     |

* $\sigma = 0.2$:

| 2.47016                              | 4     | 3.27856                                      | 4     | 2     |
| 0.02678                              | 24    | 1.75752                                      | 16    | 5     |
| 0.01667                              | 84    | 0.47349                                      | 36    | 5     |
| 0.001246                             | 217   | 0.130997                                     | 64    | 5     |

* $\sigma = 0.171$:

| 2.47016                              | 4     | 3.27856                                      | 4     | 2     |

### TABLE III. Absolute errors in Example 3.

<table>
<thead>
<tr>
<th>$|u - u_N^{(m_K)}|_{L^2(\Omega_1)}$</th>
<th>$N_1$</th>
<th>$|\phi - \phi_M^{(m_K)}|_{L^2(\Gamma_1)}$</th>
<th>$N_2$</th>
<th>$m_K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.10564</td>
<td>4</td>
<td>3.27733</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>0.07415</td>
<td>24</td>
<td>3.41856</td>
<td>16</td>
<td>5</td>
</tr>
<tr>
<td>0.005366</td>
<td>84</td>
<td>2.80817</td>
<td>36</td>
<td>5</td>
</tr>
<tr>
<td>0.000620</td>
<td>217</td>
<td>2.23380</td>
<td>64</td>
<td>5</td>
</tr>
</tbody>
</table>

* $\sigma = 0.25$:

| 1.10564                              | 4     | 3.27733                                      | 4     | 2     |
| 0.03443                              | 24    | 3.26434                                      | 16    | 5     |
| 0.006374                             | 84    | 2.50149                                      | 36    | 5     |
| 0.000560                             | 217   | 1.86517                                      | 64    | 5     |
| 0.000398                             | 475   | 1.38318                                      | 100   | 5     |

* $\sigma = 0.171$:
FIG. 3. The relative error of Example 1.

FIG. 4. The relative error of Example 2.

FIG. 5. The relative error of Example 3.
Example 1. Let the data functions be defined by

\[
F(x_1, x_2) = \frac{9}{16} A^4 \left( 2 + \frac{1}{(1 + A)^2} \right) + (2 - x_1 - x_2)^{2/3}
\]

\[
f(x_1, x_2) = \left( (2 - x_1 - x_2)^{2/3} - \frac{1}{2} \log(x_1^2 + x_2^2) \right) \bigg|_\Gamma,
\]

\[
g(x_1, x_2) = -\frac{A}{\sqrt{2}} \left( 2 + \frac{1}{1 + A} \right) (n_1 + n_2) - \frac{n_1 x_1 + n_2 x_2}{x_1^2 + x_2^2}
\]

\[
A := \frac{2\sqrt{2}}{3} (2 - x_1 - x_2)^{-1/3}.
\]

For the partition, \( \Omega_n = \{ \Omega_{ij} \} \) is a rectangle with the corners \((1 - \sigma^{i-1}, 1 - \sigma^{j-1}), (1 - \sigma^{i-1}, 1 - \sigma^j), (1 - \sigma^i, 1 - \sigma^j), 1 \leq i, j \leq n\), we use different constants \( \sigma \) and appropriate polynomial degrees. See Fig. 1 for \( \sigma = \frac{1}{2} \).

The errors of the Galerkin procedure are shown in Table I and illustrated in Fig. 3. The exact solutions are given by

\[
u_1(x_1, x_2) = (2 - x_1 - x_2)^{2/3} \quad (x \in \Omega_1)
\]

\[
u_2(x_1, x_2) = \frac{1}{2} \log(x_1^2 + x_2^2) \quad (x \in \Omega_2).
\]

Since \( u \sim r^{2/3} \), we have \( u \in B_\beta^2(\Omega_1) \) (1/3 < \( \beta < 1 \)) and \( \frac{\partial u}{r} \in B_\beta^{1/2}(\Gamma) \), (1/3 < \( \beta < 1 \)), and get exponential convergence for \( \|u - u_N\|_{\mathcal{H}^1(\Omega_1)} \) and \( \|\phi - \phi_M\|_{\mathcal{H}^{-1/2}(\Gamma)} \) [7], which is confirmed in this example. This is shown in Fig. 2 by the linear dependence of \( \log \frac{c}{\|u\|_0} \) and \( \sqrt{N_1} \) or \( \log \frac{c}{\|\phi\|_0} \) and \( \sqrt{N_2} \). Here \( \|u\|_0 := \|u\|_{\mathcal{L}^2(\Omega_1)} \) and \( \|\phi\|_0 := \|\phi\|_{\mathcal{L}^2(\Gamma)} \).

Example 2. Let the data functions be defined by

\[
F(x_1, x_2) = -\frac{32 r^2}{9 A^2} + \frac{32}{3} A + \frac{16}{3(A^{-1} + 8/3r)} - \frac{16 r^2}{9(A + 8/3r A^{-2})^2}
\]

\[
f(x_1, x_2) = \left( A^4 - \frac{1}{2} \log(x_1^2 + x_2^2) \right) \bigg|_\Gamma,
\]

\[
g(x_1, x_2) = -\frac{1}{2} n_1 + n_2 x_2 \left( \frac{16}{3} A + \frac{1}{A^{-1} + 4\sqrt{2}/3 r + 1/2r^2} \right)
\]

\[
A := (2 - x_1^2 - x_2^2)^{1/3}, \quad r := \sqrt{x_1^2 + x_2^2}.
\]

For this and the next example, we use the geometric mesh shown in Fig. 2. The corresponding errors of the Galerkin procedure are given in Table II and illustrated in Fig. 4. The exact solution is given by

\[
u_1(x_1, x_2) = (2 - x_1^2 - x_2^2)^{4/3} \quad (x \in \Omega_1)
\]

\[
u_2(x_1, x_2) = \frac{1}{2} \log(x_1^2 + x_2^2) \quad (x \in \Omega_2).
\]
Since \( u \sim r^{4/3} \), we have \( u \in B^2_H(\Omega_1) \) \((0 < \beta < 1)\) and \( \frac{\partial u}{\partial n} \in B^2_H(\Gamma) \) \((0 < \beta < 1)\), and we get exponentially fast convergence in the norms \( \|u - u_N\|_{H^1(\Omega_1)} \) and \( \|\phi - \phi_M\|_{H^{-\nu}(\Gamma)} \). This is confirmed in the numerical example, where we observe even exponential convergence of \((\phi_M)\) in \( L^2(\Gamma) \).

**Example 3.** Let the data functions be defined by

\[
\begin{align*}
F(x_1, x_2) &= -\frac{16r^2}{9A^4} + \frac{16}{3A} + \frac{8}{3(A + 4/3r)} - \frac{16r}{9(A + 4/3r)^2} + \frac{8r^2}{9(A + 4/3r)^2} \cdot A^2 + A^2, \\
f(x_1, x_2) &= \left(A^2 - \frac{1}{2} \log(x_1^2 + x_2^2)\right) \bigg|_{\Gamma}, \\
g(x_1, x_2) &= -(n_1 x_1 + n_2 x_2) \left(\frac{8}{3} A^{-1} + \frac{1}{A^{-1} + \frac{2\sqrt{2}}{3} r} + \frac{1}{2r^2}\right) \\
A &:= (2 - x_1^2 - x_2^2)^{1/3}, \quad r := \sqrt{x_1^2 + x_2^2}.
\end{align*}
\]

For this example, we also use the geometric mesh shown in Fig. 2. The corresponding errors of the Galerkin procedure are given in Table III and illustrated in Fig. 5. The exact solution is given by

\[
\begin{align*}
u_1(x_1, x_2) &= (2 - x_1^2 - x_2^2)^{2/3} \quad (x \in \Omega_1) \\
u_2(x_1, x_2) &= \frac{1}{2} \log(x_1^2 + x_2^2) \quad (x \in \Omega_2).
\end{align*}
\]

Since \( u \sim r^{4/3} \), we have \( u \in B^2_H(\Omega_1) \) \((0 < \beta < 1)\) and \( \frac{\partial u}{\partial n} \in B^2_H(\Gamma) \) \((1/6 < \beta < 1)\) that expect exponentially fast convergence in the energy norms. Numerically, we observe exponential convergence of \( \|u - u_N\|_{L^2(\Omega_1)} \) and \( \|\phi - \phi_M\|_{L^2(\Gamma)} \).

**APPENDIX**

Let \( \Omega_1 \subset R^2 \) be a bounded domain whose curvilinear boundary \( \partial \Omega_1 \) is a piecewise analytic curve \( \Gamma = \bigcup_{i=1}^{M} \Gamma_i \), where \( \Gamma_i \) is an open arc connecting the vertices \( A_i \) and \( A_{i+1}(A_{M+1} = A_1) \). Let \( \Omega_2 = R^2 \setminus \bar{\Omega}_1 \), we denote the internal angle at \( A_i \) by \( \omega_i \), and assume \( 0 < \omega_i \leq 2 \pi, 1 \leq i \leq M \). \( \partial / \partial n \) denotes the derivative with respect to the normal to \( \Gamma \) pointing from \( \Omega_1 \) to \( \Omega_2 \).

Let \( \Omega_1 \) be a bounded open set in \( R^2 \) and let \( H^k(\Omega_1), k \geq 0 \) integer, denote the usual Sobolev spaces (e.g. [19]):

\[
H^k(\Omega_1) = \left\{ u: \sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|^2_{L^2(\Omega_1)} = \|u\|^2_{H^k(\Omega_1)} < \infty \right\},
\]
where $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i \geq 0$ integers, $i = 1, 2$, $|\alpha| = \alpha_1 + \alpha_2$, and
\[
D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} u_{x_1}^{\alpha_1} u_{x_2}^{\alpha_2}.
\]
$H^{k-1/2}(\Gamma)$ is defined as the restriction of $u \in H^k(\Omega_1)$ to $\Gamma$ for integer $k \geq 1$ i.e.,
\[
H^{k-1/2}(\Gamma) = \{u|\Gamma: u \in H^k(\Omega_1)\}
\]
with
\[
\|g\|_{H^{k-1/2}(\Gamma)} = \inf_{u|\Gamma=g} \|u\|_{H^k(\Omega_1)},
\]
and for $k \leq 0$ by duality
\[
H^{k-1/2}(\Gamma) = (H^{-k+1/2}(\Gamma))'.
\]
Let \(r_i(x) = \text{dist}(x,A_i)\), and let $\beta = (\beta_1, \beta_2, \ldots, \beta_M)$ be an $M$-tuple of real numbers $0 < \beta_i < 1$. For any integer $k \geq 0$, we shall write $\beta + k = (\beta_1 + k, \beta_2 + k, \ldots, \beta_M + k)$, and $\Phi_{\beta+k}(x) = \Pi_{i=1}^M r_i^{\beta_i+k}(x)$. As in [17], we define the weighted Sobolev space for integers $k$ and $l, k \geq \beta \geq 0$, by
\[
H^{k,l}(\Omega_1) = \{u: u \in H^{k-1}(\Omega_1) \text{ if } l > 0, \|\Phi_{\beta+l-|\alpha|-l}D^\alpha u\|_{L^2(\Omega_1)} < \infty \text{ for } l \leq |\alpha| \leq k\},
\]
and the countably normed space for $l \geq 0$,
\[
B^{l}(\Omega_1) = \{u: u \in H^{k,l}(\Omega_1) \forall k \geq l, \|\Phi_{\beta+k-l}D^\alpha u\|_{L^2(\Omega_1)} \leq Cd^{k-l}(k - l)! \}
\]
for $|\alpha| = k = l, l + 1, \ldots$, with $C \geq 1, d \geq 1$ independent of $k$.

The space $H^{k-1/2,1/2-1/2}(\Gamma)$ [resp. $B^{l-1/2}(\Gamma)$] $k, l$ integers, $k \geq l \geq 1$, is the trace space of $H^{k,l}(\Omega_1)$ [resp. $B^l(\Omega_1)$], i.e., for any $g \in H^{k-1/2,1/2-1/2}(\Gamma)$ [resp. $B^{l-1/2}(\Gamma)$] there exists $G \in H^{k,l}(\Omega_1)$ [resp. $B^l(\Omega_1)$] such that $G|\Gamma = g$, and $\|g\|_{H^{k-1/2,-1/2}(\Gamma)} = \inf_{G|\Gamma = g} \|G\|_{H^{k,l}(\Omega_1)}$.

In the exterior domain $\Omega_2$, we incorporate the behavior of solutions at infinity. Let $r_i^\ast(x) = \text{min}(1, r_i(x))$ for $x \in \Omega_2$, then the weight function $\Phi_{\beta+k}(x)$ is modified by
\[
\Phi_{\beta+k}(x) = \prod_{i=1}^M (r_i^\ast(x))^{\beta_i+k}.
\]
The weighted Sobolev space, $H^{k,l}(\Omega_2)$, $k \geq l \geq 2$, is defined by
\[
H^{k,l}(\Omega_2) = \{u: u \in H^{k-1}(\Omega_2), D^\alpha u \in L^2(\Omega_2) \text{ for } 2 \leq |\alpha| < l, \|\Phi_{\beta+l-|\alpha|-l}D^\alpha u\|_{L^2(\Omega_2)} < \infty, \text{ for } l \leq |\alpha| \leq k\}.
\]

The definition of the space $B^2(\Omega_2)$ is the same as $B^2(\Omega_1)$.

The authors thank W. Thies for performing the numerical examples and the DFG for support. The work was partially supported by the DFG Forschergruppe “Zuverlässigkeit von Modellierung und Berechnung in der Angewandten Mechanik” at Hannover.
References