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Coupling of mixed finite elements and boundary elements

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The symmetric coupling of mixed finite element and boundary element methods is analysed for a model interface problem with the Laplacian. The coupling involves a further continuous *ansatz* function on the interface to link the discontinuous displacement field to the necessarily continuous boundary *ansatz* function. Quasi-optimal *a priori* error estimates and sharp *a posteriori* error estimates are established which justify adaptive mesh-refining algorithms. Numerical experiments prove the adaptive coupling as an efficient tool for the numerical treatment of transmission problems.

1. Introduction

The combination of finite element methods and boundary element methods was introduced by engineers and later mathematically justified in the 1970s with papers by Brezzi, Johnson, Nédélec, Bielak, MacCamy among others. Quasi-optimal *a priori* error estimates for the coupling of finite and boundary elements were then obtained for Lipschitz boundaries, systems of equations, and nonlinear problems (approximated by finite elements), e.g. in Gatica & Hsiao (1995) and Wendland (1988) (see also the literature quoted therein); the symmetric coupling, which is modified here, was introduced mathematically by Costabel (1987), see also Han (1990).

Automatic adaptive algorithms provide efficient discretizations if based on a rigorous *a posteriori* error analysis. For the coupling of boundary elements with the standard displacement-oriented version of finite elements, efficient and reliable *a posteriori* error bounds are derived in Carstensen (1996a) and Carstensen & Stephan (1995). It is the aim of this paper to establish reliable and efficient *a posteriori* error estimates for the coupling with mixed finite elements and so continue the work in Carstensen & Funken (1999a, b) on the coupling with nonconforming finite elements. Independent similar theoretical results for the lowest-order Raviart–Thomas element will appear in Gatica & Meddahi (1999).

Mixed methods are of particular interest in elasticity where incompressibility locking phenomena can be circumvented (cf. Brink *et al.*, 1996), or in micromagnetics (Carstensen & Funken, 1999d). We refer to Brink *et al.* (1996) for a stability and *a priori* error analysis and numerical examples in elasticity. The error indicator used in Brink *et al.* (1995) is not based on a reliable and efficient *a posteriori* error estimate in natural norms. It seems that certain jump conditions are responsible which also arise in a corresponding Laplace problem (Braess & Verfürth, 1996). This motivates our investigations in the coupling of mixed finite element methods and conform boundary element methods for the Laplace

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problem. Ongoing research will cover robust reliable and efficient error control in elasticity problems based on the techniques presented here (Carstensen & Funken, 1999c).

In this paper we analyse a model problem (cf. Carstensen, 1996a; Carstensen & Funken, 1999a; Gatica & Hsiao, 1995; Wendland, 1988) which involves the Laplacian in a bounded two-dimensional Lipschitz domain Ω with boundary $\Gamma = \partial\Omega$ and exterior domain $\Omega_c := \mathbb{R}^2 \setminus \overline{\Omega}$. Given jump conditions $u_0 \in H^1(\Gamma)$, $t_0 \in L^2(\Gamma)$ and a right-hand side $f \in L^2(\Omega)$, we seek functions $u \in H^1(\Omega)$, $u_c \in H^1_{loc}(\Omega_c)$ and real constants a and b satisfying

$$-\Delta u = f \qquad \text{in } \Omega, \tag{1.1}$$

$$\Delta u_c = 0 \qquad \text{in } \Omega_c, \tag{1.2}$$

$$\lim_{|x|\to\infty} \{u_c(x) - b\ \log(x)\} = a, \tag{1.3}$$

$$u = u_c + u_0 \qquad \text{on } \Gamma, \tag{1.4}$$

$$\partial u/\partial n = \partial u_c/\partial n + t_0 \quad \text{on } \Gamma. \tag{1.5}$$

Here, Δ denotes the Laplacian and n is the exterior unit normal on Ω.

It is known that the interface problem (1.1)–(1.5) has a unique solution if we specify $a = 0$ (see, e.g., Carstensen, 1996a; Carstensen & Funken, 1999a; Gatica & Hsiao, 1995; Wendland, 1988). In the mixed formulation in Ω we split equation (1.1) into

$$p = \nabla u \qquad \text{in } \Omega, \tag{1.6}$$

$$-\text{div}\, p = f \qquad \text{in } \Omega, \tag{1.7}$$

and recast condition (1.6) using integration by parts. The equivalent weak form obtained in Section 2 reads: *Seek* $(p, u, \xi) \in H(\text{div};\Omega) \times L^2(\Omega) \times H_0^{1/2}(\Gamma)$ *such that for all* $(q, v, \eta) \in H(\text{div};\Omega) \times L^2(\Omega) \times H_0^{1/2}(\Gamma)$

$$a(p, \xi; q, \eta) + b(u; q, \eta) = \langle g_1, q \cdot n\rangle + \langle g_2, \eta\rangle, \tag{1.8}$$

$$b(v; p, \xi) = -(f, v), \tag{1.9}$$

where we are given data $g_1 := u_0 + \frac{1}{2}\mathcal{V}t_0 \in H^{1/2}(\Gamma)$, $g_2 := \frac{1}{2}(\mathcal{K}^* + 1)t_0 \in H^{-1/2}(\Gamma)$, and $f \in L^2(\Omega)$, and bilinear forms a and b defined by

$$\begin{aligned} a(p, \xi; q, \eta) &:= (p, q) + \tfrac{1}{2}\langle \mathcal{V}(p \cdot n) - (\mathcal{K} + 1)\xi, q \cdot n\rangle \\ &\quad + \tfrac{1}{2}\langle \mathcal{W}\xi + (\mathcal{K}^* + 1)(p \cdot n), \eta\rangle, \end{aligned} \tag{1.10}$$

$$b(u; q, \eta) := (u, \text{div}\, q), \tag{1.11}$$

for $p, q \in H(\text{div};\Omega)$, $u \in L^2(\Omega)$, $\xi, \eta \in H_0^{1/2}(\Gamma) \equiv H^{1/2}(\Gamma)/\mathbb{R}$, and with certain boundary integral operators and Sobolev spaces (described in Section 2). The $L^2(\Omega)$ scalar product is written as $(\cdot, \cdot)$ while $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^s(\Gamma)$ and $H^{-s}(\Gamma)$ (defined by extending the scalar product in $L^2(\Gamma)$). We remark that $\xi := u_c|_\Gamma$.

The discretization of (1.8)–(1.9) consists essentially in replacing the above Sobolev spaces by finite dimensional subspaces $\mathcal{M} \subset H(\text{div};\Omega)$, $\mathcal{L} \subset L^2(\Omega)$, and $\mathcal{S} \subset H_0^{1/2}(\Gamma)$ and so involves finite element spaces $\mathcal{M}$ named after Raviart–Thomas, Brezzi–Douglas–Marini and Brezzi–Douglas–Fortin–Marini.

A complete *a priori* and *a posteriori* error analysis is presented in this paper, which is organized as follows. The Sobolev spaces and the related boundary integral operators are recalled from the literature with their relevant mapping properties in Section 2. We also quote some basic facts about the representation formula which is required to recast the exterior part of the interface problem and to establish the mixed weak formulation (1.8)–(1.9). The discretization is described in Section 3 where quasi-optimal convergence is shown in an *a priori* error analysis. An *a posteriori* error analysis is given in Section 4 which provides a reliable and efficient computable error bound. The proof is based on a Helmholtz-decomposition as in Alonso (1996) and Carstensen (1997a), but here we omit orthogonality: the interface conveys Dirichlet and Neumann conditions simultaneously and so additional considerations are necessary that rely on the positive definiteness of the single-layer potential and hypersingular integral operator. The upper error bound can be evaluated elementwise and so serves as an error indicator in an adaptive mesh-refining algorithm proposed in Section 5, where we also sketch our numerical implementation. Numerical examples are reported in Section 6 which confirm our theoretical convergence results and illustrate the practical performance of the scheme.

We finally stress that the model situation could be generalized to other operators, e.g. to inhomogeneous elliptic operators such as linear elasticity (Brink *et al.*, 1996), or other dimensions (with adopted radiation conditions (1.3)). Moreover, we might add Dirichlet, Neumann or mixed boundary conditions or further right-hand sides.

2. Preliminaries

Let $H^s(\Omega)$ denote the usual Sobolev spaces (Lions & Magenes, 1972) with the trace spaces $H^{s-1/2}(\Gamma)$ ($s \in \mathbb{R}$) for a bounded Lipschitz domain Ω with boundary Γ. Let $\|\cdot\|_{H^k(\omega)}$ and $|\cdot|_{H^k(\omega)}$ denote the norm and semi-norm in $H^k(\omega)$ for $\omega \subseteq \Omega$ and an integer k. The space

$$H(\mathrm{div};\Omega) := \{q \in L^2(\Omega)^2 : \mathrm{div}\, q \in L^2(\Omega)\}$$

is equipped by its natural norm

$$\|\cdot\|_{H(\mathrm{div};\Omega)} := (\|\cdot\|^2_{L^2(\Omega)} + \|\mathrm{div}\cdot\|^2_{L^2(\Omega)})^{1/2}.$$

Given $v \in H^{1/2}(\Gamma)$ and $\phi \in H^{-1/2}(\Gamma)$, the boundary integral operators in (1.10)–(1.11) are defined, for $z \in \Gamma$, by

$$(\mathcal{V}\phi)(z) := -\frac{1}{\pi}\int_\Gamma \phi(\zeta)\log|z-\zeta|\,\mathrm{d}s_\zeta,$$
$$(\mathcal{K}v)(z) := -\frac{1}{\pi}\int_\Gamma v(\zeta)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\,\mathrm{d}s_\zeta,$$

$$(\mathcal{K}^*\phi)(z) := -\frac{1}{\pi}\int_\Gamma \phi(\zeta)\frac{\partial}{\partial n_z}\log|z-\zeta|\,\mathrm{d}s_\zeta,$$
$$(\mathcal{W}v)(z) := \frac{1}{\pi}\frac{\partial}{\partial n_z}\int_\Gamma v(\zeta)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\,\mathrm{d}s_\zeta.$$

The linear boundary integral operators are continuous when mapping between the following Sobolev spaces

$$\begin{aligned}\mathcal{V} &: H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma),\\ \mathcal{K} &: H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma),\\ \mathcal{K}^* &: H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma),\\ \mathcal{W} &: H^{s+1/2}(\Gamma) \to H^{s-1/2}(\Gamma),\end{aligned}$$

where $s \in [-\frac{1}{2}, \frac{1}{2}]$ (Costabel, 1988). The single-layer potential $\mathcal{V}$ is symmetric, the double-layer potential $\mathcal{K}$ has the dual $\mathcal{K}^*$ and the hypersingular operator $\mathcal{W}$ is symmetric. Both $\mathcal{V}$ and $\mathcal{W}$ are strongly elliptic in the sense that they satisfy a Gårding inequality (in the above spaces with $s = 0$) (Costabel, 1988).

Let $H_0^s(\Gamma) := \{\phi \in H^s(\Gamma) : \langle 1, \phi\rangle = 0\} \equiv H^s(\Gamma)/\mathbb{R}$. Then, it is known that $\mathcal{V} : H_0^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $\mathcal{W} : H_0^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ are positive definite. Assuming that the capacity of Γ is smaller than one, the single-layer potential $\mathcal{V}$ is positive definite on $H^{-1/2}(\Gamma)$. See, e.g., Costabel & Stephan (1985), Costabel (1988), Gaier (1976), Sloan & Spence (1988), Stephan & Wendland (1976) and Stephan *et al.* (1979) for more details.

There is an infinite set of formulae which characterize the Cauchy data $(u_c, \partial u_c/\partial n)|_\Gamma$ of a function u_c with (1.2)–(1.3) and we quote only one from the literature.

LEMMA 1 (Costabel & Stephan, 1985) Let $u_c \in H^1_{loc}(\Omega_c)$ satisfy (1.2) and (1.3), then $(\xi, \phi) := (u_c, \partial u_c/\partial n)|_\Gamma \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ satisfies

$$2\begin{pmatrix}\xi\\ \phi\end{pmatrix} = \begin{pmatrix}1+\mathcal{K} & -\mathcal{V}\\ -\mathcal{W} & 1-\mathcal{K}^*\end{pmatrix}\begin{pmatrix}\xi\\ \phi\end{pmatrix} + \begin{pmatrix}2a\\ 0\end{pmatrix}. \tag{2.1}$$

Conversely, for each $(\xi, \phi) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ there exists a function $u_c \in H^1_{loc}(\Omega_c)$ with (1.2)–(1.3) if and only if (2.1) holds. The function u_c is given by the representation formula, for $x \in \Omega_c$,

$$u_c(x) = \frac{1}{2\pi}\int_\Gamma \phi(z)\,\log|x-z|\,\mathrm{d}s_z - \frac{1}{2\pi}\int_\Gamma \xi(z)\,\frac{\partial}{\partial n_z}\log|x-z|\,\mathrm{d}s_z + a. \tag{2.2}$$

The problem (1.1)–(1.5) has a unique solution and so the equivalent problem (1.8)–(1.9) has a unique solution also.

To our knowledge the following result is not available in this precise form. For related modifications we refer to Brink *et al.* (1996), Cartensen & Funken (1999a) and Gatica & Hsiao (1995) and the references quoted therein.

THEOREM 1 The interface problem (1.1)–(1.5) and the weak formulation (1.8)–(1.9) are formally equivalent: If (u, u_c) solves (1.1)–(1.5) then $p = \nabla u$, u, and $\xi = u_c|_\Gamma$ solve (1.8)–(1.9). If (p, u, ξ) solves (1.8)–(1.9), then, given $\phi = \partial u/\partial n - t_0$, (2.2) defines a function u_c such that (u, u_c) solves (1.1)–(1.5).

Proof. The Cauchy data $(u_c, \partial u_c/\partial n)|_\Gamma =: (\xi, \phi)$ of a function u_c which satisfies (1.2)–(1.3) with $a = 0$ are characterized in Lemma 1 to satisfy (2.1), namely

$$2u_c|_\Gamma = (1+\mathcal{K})\xi - \mathcal{V}(p\cdot n - t_0), \tag{2.3}$$

$$0 = \mathcal{W}\xi + (1+\mathcal{K}^*)(p\cdot n - t_0). \tag{2.4}$$

Note that $\xi = u_c|_\Gamma = u|_\Gamma - u_0$ and $\phi = \partial u_c/\partial n|_\Gamma = \partial u/\partial n|_\Gamma - t_0 = p \cdot n - t_0$.

Multiplying (1.6) by $q \in H(\mathrm{div};\Omega)$ and integrating by parts we obtain

$$(p, q) + (\mathrm{div}\, q, u) = \langle u|_\Gamma, q \cdot n\rangle = \langle u_c|_\Gamma + u_0, q \cdot n\rangle. \tag{2.5}$$

Substitution of u_c by (2.3) shows (1.8) for $\eta = 0$. The weak form of (2.4) gives (1.8) for $q = 0$ and arbitrary $\eta \in H^{-1/2}(\Gamma)$. Finally, the weak form of (1.7) is (1.9).

Notice that $\mathcal{W}1 = 0 = (1 + \mathcal{K})1$ (proved by (2.1) for $(\xi, \phi) = (1, 0)$ and $a = 1$). Thus the variable ξ is determined in (2.3)–(2.4) up to an additive constant and we fix this constant by $\langle \xi, 1\rangle = 0$, i.e. $\xi \in H_0^{1/2}(\Gamma)$. ($u_c$ is unique because of $a = 0$ while ξ acts as a layer in the boundary integral operators and is non-unique, but $\xi - u_c|_\Gamma$ is constant.)

The preceeding calculations establish (1.8)–(1.9) and the same arguments yield the reverse implication and so prove equivalence. □

3. Discrete problem and *a priori* error analysis

Assume that the triangulation $\mathcal{T}$ of the domain Ω with polygonal boundary Γ is regular in the sense of Ciarlet (cf. Brenner & Scott, 1994; Ciarlet, 1978) and that each $T \in \mathcal{T}$ is a closed triangle with interior angles greater than the (universal) constant $c_\theta > 0$ and diameter $h_T > 0$. On the boundary Γ there is a mesh $\mathcal{G} := \{E \in \mathcal{E} : E \subset \Gamma\}$ induced by the set of edges $\mathcal{E}$ of triangles in $\mathcal{T}$. The length of an edge $E \in \mathcal{E}$ is $h_E := \mathrm{diam}(E)$. On the boundary Γ, we consider continuous *ansatz* functions that include the $\mathcal{G}$-piecewise affines, i.e.,

$$\mathcal{S}^1(\mathcal{G}) := \{w \in C(\Gamma) : \forall\, E \in \mathcal{G},\ w|_E \text{ affine}\}, \tag{3.1}$$

$$\mathcal{S} := \mathcal{S}^1(\mathcal{G})/\mathbb{R} := \{w \in \mathcal{S}^1(\mathcal{G}) : \langle w, 1\rangle = 0\} \subset H_0^{1/2}(\Gamma). \tag{3.2}$$

Let $\mathcal{L} \subseteq L^2(\Omega)$ and $\mathcal{M} \subseteq H(\mathrm{div};\Omega)$ be finite element spaces subordinated to $\mathcal{T}$ (Brezzi & Fortin, 1991) which satisfy the LBB-condition, i.e.,

$$\inf_{V \in \mathcal{L}\setminus\{0\}} \sup_{Q \in \mathcal{M}\setminus\{0\}} \frac{(\mathrm{div}\, Q, V)}{\|Q\|_{H(\mathrm{div};\Omega)} \|V\|_{L^2(\Omega)}} \geqslant \beta > 0. \tag{3.3}$$

For each $V \in \mathcal{L}$, $Q \in \mathcal{M}$, and $T \in \mathcal{T}$ we suppose that $V|_T$ and $Q|_T$ are polynomials and that $\mathcal{L}$ includes $\mathcal{T}$-piecewise constant functions. Then the discrete interface problem reads: *Seek* $(P, U, \Xi) \in \mathcal{M} \times \mathcal{L} \times \mathcal{S}$ *satisfying, for all* $(Q, V, \Theta) \in \mathcal{M} \times \mathcal{L} \times \mathcal{S}$,

$$a(P, \Xi; Q, \Theta) + b(U; Q, \Theta) = \langle \tilde{g}_1, Q \cdot n\rangle + \langle \tilde{g}_2, \Theta\rangle, \tag{3.4}$$

$$b(V; P, \Xi) = -(f, V). \tag{3.5}$$

Here, $\tilde{g}_1 := u_0 + \frac{1}{2}\mathcal{V}\tilde{t}_0 \in H^{1/2}(\Gamma)$ and $\tilde{g}_2 := \frac{1}{2}(\mathcal{K}^* + 1)\tilde{t}_0 \in H^{-1/2}(\Gamma)$ for some approximation $\tilde{t}_0 \in H_0^{-1/2}(\Gamma)$ to t_0 (e.g., the $\mathcal{G}$-piecewise integral mean of $t_0 \in L^2(\Gamma)$).

THEOREM 2 There exists a constant C which depends only on β in (3.3) and on Ω such that we have

$$\|p - P\|_{H(\mathrm{div};\Omega)} + \|u - U\|_{L^2(\Gamma)} + \|\xi - \Xi\|_{H^{1/2}(\Gamma)}$$
$$\leqslant C \left\{ \inf_{Q \in \mathcal{M}} \|p - Q\|_{H(\mathrm{div};\Omega)} + \inf_{\mathcal{L} \in \mathcal{L}} \|u - V\|_{L^2(\Omega)} \right.$$
$$\left. + \inf_{\Theta \in \mathcal{S}} \|\xi - \Theta\|_{H_0^{1/2}(\Gamma)} + \|t_0 - \tilde{t}_0\|_{H^{-1/2}(\Gamma)} \right\}. \quad (3.6)$$

Proof. Let $(\tilde{p}, \tilde{u}, \tilde{\xi}) \in \mathcal{M} \times \mathcal{L} \times \mathcal{S}$ approximate the exact solution (p, u, ξ) and let Z denote the kernel of b, i.e., $Z := \{q \in H(\mathrm{div};\Omega) : \mathrm{div}\, q = 0\} \times H_0^{1/2}(\Gamma)$. The bilinear form a is Z-elliptic according to

$$a(p, \xi; p, \xi) = \|p\|^2_{L^2(\Omega)} + \tfrac{1}{2}\langle \mathcal{V}(p \cdot n), p \cdot n\rangle + \tfrac{1}{2}\langle \mathcal{W}\xi, \xi\rangle, \quad (3.7)$$

for all $p \in H(\mathrm{div};\Omega)$ and $\xi \in H^{1/2}(\Gamma)$, and the positive definiteness of $\mathcal{V}$ and $\mathcal{W}$. Because of this and (3.3), we conclude with the theory of mixed finite element schemes (Brezzi & Fortin, 1991) that there exist $(Q, V, \Theta) \in \mathcal{M} \times \mathcal{L} \times \mathcal{S}$ with $\|Q\|_{H(\mathrm{div};\Omega)} + \|V\|_{L^2(\Omega)} + \|\Theta\|_{H^{1/2}(\Gamma)} \leqslant 1$ and

$$C(\Omega, \beta)\{\|\tilde{p} - P\|_{H(\mathrm{div};\Omega)} + \|\tilde{u} - U\|_{L^2(\Gamma)} + \|\tilde{\xi} - \Xi\|_{H_0^{1/2}(\Gamma)}\}$$
$$\leqslant a(\tilde{p} - P, \tilde{\xi} - \Xi; Q, \Theta) + b(\tilde{u} - U; Q, \Theta) + b(V; \tilde{p} - P, \tilde{\xi} - \Xi)$$
$$= a(p - P, \xi - \Xi; Q, \Theta) + b(u - U; Q, \Theta) + b(V; p - P, \xi - \Xi)$$
$$+ a(\tilde{p} - p, \tilde{\xi} - \xi; Q, \Theta) + b(\tilde{u} - u; Q, \Theta) + b(V; \tilde{p} - p, \tilde{\xi} - \xi). \quad (3.8)$$

The constant $C(\Omega, \beta) > 0$ depends on β and the norms of a and b. By the definition of $\tilde{g}_j$, there is a constant c_1 that depends on the norms of $\mathcal{V}$ and $\mathcal{K}$ with

$$\langle g_1 - \tilde{g}_1, Q \cdot n\rangle + \langle g_2 - \tilde{g}_2, \Theta\rangle \leqslant c_1 \|t_0 - \tilde{t}_0\|_{H^{-1/2}(\Gamma)}. \quad (3.9)$$

The last line of (3.8) is bounded by the right-hand side of (3.6) and the penultimate line is equal to the left-hand side of (3.9). From this resulting estimate and the triangle inequality, we conclude the proof. □

4. A reliable and efficient *a posteriori* error estimate

Let $(u, u_c) \in H^1(\Omega) \times H^1_{loc}(\Omega_c)$ solve (1.1)–(1.5) and define $p := \nabla u$, and $\xi := u|_\Gamma - u_0 = u_c|_\Gamma$. Given a solution (P, U, Ξ) to (3.4)–(3.5), define $J_\tau \in L^2(\cup\mathcal{E})$ on each edge $E \in \mathcal{E}$ by

$$J_\tau|_E := \begin{cases} [P \cdot \tau_E] & \text{if } E \not\subset \Gamma, \\ 2P \cdot \tau_E - \partial/\partial s(2u_0 + (\mathcal{K} + 1)\Xi - \mathcal{V}(P \cdot n - \tilde{t}_0)) & \text{if } E \subset \Gamma. \end{cases} \quad (4.1)$$

Here, n_E denotes the normal and τ_E the tangential unit vector along the edge E, the square brackets denote the jump of the piecewise Lipschitz continuous quantities.

TABLE 1

Element	$\mathcal{M}\|_T$	$\mathcal{L}\|_T$
RT	$\mathbb{P}_k^2 \times x \cdot \mathbb{P}_k$	$\mathbb{P}_k$
BDM	$\mathbb{P}_{k+1}^2$	$\mathbb{P}_k$
BDFM	$\{q \in \mathbb{P}_k^2 \mid (q \cdot n)\|_E \in \mathbb{P}_k(E),\, E \subseteq \partial T\}$	$\mathbb{P}_k$

As a key observation in mixed finite element methods, e.g., for Raviart–Thomas elements, Brezzi–Douglas–Fortin elements, or Brezzi–Douglas-Fortin–Marini elements (cf. Table 1 and Brezzi & Fortin, 1991 for details) we have (see Carstensen, 1997a)

$$\operatorname{Curl} B \in \mathcal{M} \text{ for all } \mathcal{T}\text{-piecewise affine } B \in C(\Omega). \tag{4.2}$$

THEOREM 3 Suppose Ω is simply connected and assume (4.2). Then there exists a positive constant C which depends only on c_θ and Ω, such that there holds

$$\begin{aligned}
&\|p - P\|_{L^2(\Omega)}^2 + \|p \cdot n - P \cdot n\|_{H^{-1/2}(\Gamma)}^2 + \|\xi - \Xi\|_{H_0^{1/2}(\Gamma)/\mathbb{R}}^2 \\
&\leqslant C \left\{ \sum_{T \in \mathcal{T}} h_T^2 \int_T (|\operatorname{curl} P|^2 + |f + \operatorname{div} P|^2)\, \mathrm{d}x + \sum_{E \in \mathcal{E}} h_E \|J_\tau\|_{L^2(E)}^2 \right. \\
&\quad \left. + \sum_{E \in \mathcal{G}} h_E \|\mathcal{W}\Xi + (\mathcal{K}^* + 1)(P \cdot n - \tilde{t}_0)\|_{L^2(E)}^2 + \|t_0 - \tilde{t}_0\|_{H^{-1/2}(\Gamma)}^2 \right\}.
\end{aligned} \tag{4.3}$$

If, in addition, Ω is H^2-regular (e.g. Ω convex or Γ is C^2) and $(\mathcal{M}, \mathcal{L})$ are the discrete spaces obtained from Raviart–Thomas elements or Brezzi–Douglas–Fortin–Marini elements, then

$$\begin{aligned}
\|u - U\|_{L^2(\Omega)}^2 \leqslant C \Bigg\{ &\|p \cdot n - P \cdot n\|_{H^{-1/2}(\Gamma)}^2 \\
&+ \sum_{T \in \mathcal{T}} h_T^2 \int_T (|P - \nabla V|^2 + |f + \operatorname{div} P|^2)\, \mathrm{d}x \Bigg\},
\end{aligned} \tag{4.4}$$

where V is an arbitrary element in $\mathcal{L}$ (possibly $V = U$).

REMARK 1 The proof of *a posteriori* estimates for BDM finite elements involves a further approximation error since property (4.35) below does not hold for these elements. For simplicity it is not presented in this paper.

REMARK 2 As shown in the proof below (cf. equation (4.24)), $\|t_0 - \tilde{t}_0\|_{H^{1/2}(\Gamma)}$ can be replaced by $c_6 \|h_{\mathcal{G}}(t_0 - \tilde{t}_0)\|_{L^2(\Gamma)}$ if $\tilde{t}_0$ is the $\mathcal{G}$-piecewise integral mean of t_0 and $h_{\mathcal{G}}$ is the $\mathcal{G}$-piecewise constant mesh-size, $h_{\mathcal{G}}|_E = h_E$ for $E \in \mathcal{G}$. Furthermore, if t_0 is $\mathcal{G}$-piecewise smooth, $\|h_{\mathcal{G}}(t_0 - \tilde{t}_0)\|_{L^2(\Gamma)} = O(h_{\max}^{3/2})$ which is a higher-order approximation term since we expect at most linear convergence for the lowest-order schemes. Thus, we could generically neglect this higher-order contribution.

REMARK 3 The use of $\tilde{t}_0$ as a piecewise constant simplifies the calculation of $\langle \mathcal{V}t_0, q \cdot n\rangle$, $\langle K^* t_0, \eta\rangle$ in (3.4) and the pointwise evaluation of the integral operators $\mathcal{V}t_0$, $K^* t_0$ in the *a posteriori* estimate (4.3). These terms can be calculated analytically for piecewise polynomials (Carstensen & Funken, 1999a, b).

REMARK 4 For f smooth, $\|f + \operatorname{div} P\|_{L^2(T)} \leqslant h_T \|\nabla f\|_{L^2(T)}$ according to a Poincaré inequality (div P is constant and $f + \operatorname{div} P$ has integral mean 0). Hence, for the lowest-order schemes we could generically neglect the contribution $h_T\|f + \operatorname{div} P\|_{L^2(T)} = O(h_T^2)$.

REMARK 5 Hence, for smooth data f and t_0 and RT finite elements ($k = 0$), the error indicator consists of edge contributions only.

Proof of Theorem 3. For simplicity, set $d := p - P$, $e := u - U$, and $\delta := \xi - \Xi$. The local mesh-sizes $h_{\mathcal{T}} \in L^\infty(\Omega)$ and $h_{\mathcal{E}} \in L^\infty(\cup\mathcal{E})$ are piecewise constant functions with $(h_{\mathcal{T}})|_T := h_T := \operatorname{diam}(T)$, $T \in \mathcal{T}$, and $(h_{\mathcal{E}})|_E := h_E := \operatorname{diam}(E)$, $E \in \mathcal{E}$. According to (1.8), (2.4), and (3.4)

$$\begin{aligned} \varrho &:= \mathcal{W}\delta + (\mathcal{K}^* + 1)(d \cdot n) + 2(\tilde{g}_2 - g_2) \perp \mathcal{S}^1(\mathcal{G}) \\ &= -\mathcal{W}\Xi - (\mathcal{K}^* + 1)(P \cdot n - \tilde{t}_0) \end{aligned} \tag{4.5}$$

with $\perp$ denoting orthogonality in $L^2(\Gamma)$. (Note that $\langle \varrho, 1\rangle = 0$ by $\mathcal{W}1 = 0$ and $\mathcal{K}1 = -1$.) Define a function $g \in H^1(\Omega)$, for $z \in \Omega$,

$$g(z) := -\frac{1}{2\pi}\int_\Gamma \delta(\zeta)\frac{\partial}{\partial n_\zeta}\log|z - \zeta|\,\mathrm{d}s_\zeta + \frac{1}{2\pi}\int_\Gamma (d \cdot n)(\zeta)\log|z - \zeta|\,\mathrm{d}s_\zeta, \tag{4.6}$$

with trace and trace estimate (according to the mapping properties of the single- and double-layer potential operators)

$$g|_\Gamma = \tfrac{1}{2}\{(\mathcal{K} + 1)\delta - \mathcal{V}(d \cdot n)\} \in H^{1/2}(\Gamma), \tag{4.7}$$

$$\|g\|_{H^{1/2}(\Gamma)} \leqslant c_2(\|d \cdot n\|_{H^{-1/2}(\Gamma)} + \|\delta\|_{H^{1/2}(\Gamma)}). \tag{4.8}$$

The constant c_2 as well as the constants $c_3, \ldots, c_{16}$ throughout this proof depend on Ω, Γ and c_θ only.

Let $\alpha \in H^1(\Omega)$ be the unique solution of the Dirichlet problem

$$\Delta\alpha = \operatorname{div} d \quad \text{in } \mathcal{D}'(\Omega) \text{ and } \alpha|_\Gamma = g|_\Gamma. \tag{4.9}$$

Then $d - \nabla\alpha$ is divergence free and, since Ω is simply connected, there exists a function $\beta \in H^1(\Omega)/\mathbb{R} := \{w \in H^1(\Omega) : \int_\Omega w\,\mathrm{d}x = 0\}$ with

$$d = \nabla\alpha + \operatorname{Curl}\beta. \tag{4.10}$$

Throughout this paper, we define

$$\operatorname{curl} a = a_{2,1} - a_{1,2} \quad \text{resp.} \quad \operatorname{Curl} b = \begin{pmatrix} b_{,2} \\ -b_{,1} \end{pmatrix}$$

for a vector a and a scalar b.

The function β in (4.10) can be characterized by a Neumann problem and so we may prescribe

$$\frac{\partial \beta}{\partial n} = d \cdot t - \frac{\partial g}{\partial s} \tag{4.11}$$

where $\partial g/\partial s \in H^{-1/2}(\Gamma)$ is the derivative of $g|_\Gamma$ along Γ with respect to the arc length while $d \cdot t$ is defined in a weak sense according to Stokes theorem. We refer to Girault & Raviart (1986) for details and proofs and mention

$$\begin{aligned} \|\nabla \alpha\|_{L^2(\Omega)} + \|\mathrm{Curl}\,\beta\|_{L^2(\Omega)} &\leqslant c_3\{\|d\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}\} \\ &\leqslant c_4\{\|d\|_{L^2(\Omega)} + \|\delta\|_{H^{1/2}(\Gamma)} + \|d \cdot n\|_{H^{-1/2}(\Gamma)}\}. \end{aligned} \tag{4.12}$$

By (4.9), (4.10), and an integration by parts, we deduce

$$\|d\|^2_{L^2(\Omega)} = -(\alpha, \mathrm{div}\, d) + \langle g, d \cdot n\rangle + (\mathrm{Curl}\,\beta, \nabla u) - (P, \mathrm{Curl}\,\beta). \tag{4.13}$$

Since (1.9) and (3.5) imply that $(\alpha, \mathrm{div}\, d) = (\alpha - A, \mathrm{div}\, d)$ for any $A \in \mathcal{L}$, it follows from a Poincaré inequality that

$$-(\alpha, \mathrm{div}\, d) = (A - \alpha, \mathrm{div}\, d) \leqslant c_5 \|\nabla \alpha\|_{L^2(\Omega)}\, \|h_{\mathcal{T}}(f + \mathrm{div}\, P)\|_{L^2(\Omega)}. \tag{4.14}$$

Let B be a continuous $\mathcal{T}$-piecewise affine approximation to β, e.g. the Clément interpolation (Clément, 1975; Verfürth, 1996). By assumption (4.2), $\mathrm{Curl}\, B \in \mathcal{M}$, and because of $\mathrm{div}\,\mathrm{Curl}\, B = 0$, (3.4) yields

$$(P, \mathrm{Curl}\, B) = \tfrac{1}{2}\langle 2\tilde{g}_1 - \mathcal{V}(P \cdot n) + (\mathcal{K} + 1)\Xi, \partial B/\partial s\rangle. \tag{4.15}$$

Then, an integration by parts leads to

$$\begin{aligned} (\mathrm{Curl}\,\beta, \nabla u) - (P, \mathrm{Curl}\,\beta) &= (P, \mathrm{Curl}\,(B - \beta)) \\ &\quad - \tfrac{1}{2}\langle 2\tilde{g}_1 - \mathcal{V}(P \cdot n) + (\mathcal{K} + 1)\Xi, \partial B/\partial s\rangle + \langle u, \partial\beta/\partial s\rangle. \end{aligned} \tag{4.16}$$

An elementwise integration by parts of the first term on the right-hand side of (4.16) shows that (4.16) equals (writing $\cup\mathcal{E} \setminus \Gamma$ for the union of inner edges $(\cup\mathcal{E}) \setminus \Gamma$)

$$\begin{aligned} &-(\beta - B, \mathrm{curl}_{\mathcal{T}} P) + \int_{\cup\mathcal{E}\setminus\Gamma} [P \cdot t](\beta - B)\, \mathrm{d}s \\ &\quad + \langle P \cdot t, \beta - B\rangle \tfrac{1}{2}\langle \partial/\partial s(2\tilde{g}_1 - \mathcal{V}(P \cdot n) + (\mathcal{K} + 1)\Xi), B\rangle - \langle \partial u/\partial s, \beta\rangle. \end{aligned} \tag{4.17}$$

In the last terms, we integrated by parts on Γ, i.e. $\langle a, \partial b/\partial s\rangle = -\langle \partial a/\partial s, b\rangle$ for $a, b \in H^1(\Gamma)$ (note that the functions u_0, $\mathcal{V}\Phi$, $(\mathcal{K} + 1)\Xi$, etc, belong to $H^1(\Gamma)$). The terms on Γ in (4.17) can be recast with (1.4), (2.3), (4.1), (4.7), and the definition of $\tilde{g}_1$ into

$$\frac{1}{2}\langle J_\tau, \beta - B\rangle - \left\langle \beta, \frac{\partial}{\partial s} g\right\rangle + \frac{1}{2}\left\langle \beta \frac{\partial}{\partial s}\mathcal{V}(\tilde{t}_0 - t_0)\right\rangle. \tag{4.18}$$

Integrating by parts on Γ again, we deduce from (4.16)–(4.18) the identity

$$(\operatorname{Curl}\beta, \nabla u) - (P, \operatorname{Curl}\beta) = -(\beta - B, \operatorname{curl}_{\mathcal{T}} P) + \langle \partial\beta/\partial s, g\rangle + \int_{\cup\mathcal{E}\setminus\Gamma} [P \cdot t](\beta - B)\,\mathrm{d}s + \frac{1}{2}\langle J_\tau, \beta - B\rangle + \frac{1}{2}\left\langle \beta \frac{\partial}{\partial s}\mathcal{V}(\tilde{t}_0 - t_0)\right\rangle. \quad (4.19)$$

According to (4.10), we have

$$\langle \partial\beta/\partial s, g\rangle = \langle g, \operatorname{Curl}\beta \cdot n\rangle = \langle g, d \cdot n\rangle - \langle g, \partial\alpha/\partial n\rangle. \quad (4.20)$$

Since $g = \alpha$ on Γ we infer with Green's formula

$$\|\nabla\alpha\|^2_{L^2(\Omega)} - \langle g, \partial\alpha/\partial n\rangle = -(\alpha, \Delta\alpha) = -(\alpha, \operatorname{div} d) \leqslant c_5 \|\nabla\alpha\|_{L^2(\Omega)} \|h_{\mathcal{T}}(f + \operatorname{div} P)\|_{L^2(\Omega)} \quad (4.21)$$

as in (4.14). According to (4.5), (4.7), and since $(\mathcal{K}^* + 1)$ is dual to $(\mathcal{K} + 1)$,

$$2\langle g, d \cdot n\rangle = \langle (\mathcal{K} + 1)\delta - \mathcal{V}(d \cdot n), d \cdot n\rangle = -\langle \mathcal{V}(d \cdot n), d \cdot n\rangle - \langle \mathcal{W}\delta, \delta\rangle + \langle \varrho + 2(g_2 - \tilde{g}_2), \delta\rangle. \quad (4.22)$$

We quote from Theorem 2 of Carstensen (1997b) that $\varrho \in L^2(\Gamma)$ being $L^2(\Gamma)$-orthogonal to continuous and $\mathcal{G}$-piecewise affine functions (according to (4.5)) is sufficient for the estimate

$$\|\varrho\|_{H^{-1/2}(\Gamma)} \leqslant c_6 \|h_{\mathcal{E}}^{1/2} \varrho\|_{L^2(\Gamma)}. \quad (4.23)$$

(The constant c_6 depends weakly on the ratio of two neighbouring edges along Γ and so is bounded in terms of c_θ.) Similarly, if $\tilde{t}_0$ is the $\mathcal{G}$-piecewise integral mean of t_0, we have

$$\|t_0 - \tilde{t}_0\|_{H^{1/2}(\Gamma)} \leqslant c_6 \|h_{\mathcal{G}}^{1/2}(t_0 - \tilde{t}_0)\|_{L^2(\Gamma)}. \quad (4.24)$$

Gathering (4.13), (4.14), (4.19)–(4.21), and (4.23) together, we obtain with Cauchy's inequality

$$\begin{aligned}\|d\|^2_{L^2(\Omega)} + \|\nabla\alpha\|^2_{L^2(\Omega)} &+ \langle \mathcal{V}(d \cdot n), d \cdot n\rangle + \langle \mathcal{W}\delta, \delta\rangle \\ \leqslant c_7 \Big\{ &\|\nabla\alpha\|_{L^2(\Omega)} \|h_{\mathcal{T}}(f + \operatorname{div} P)\|_{L^2(\Omega)} \\ &+ \|\delta\|_{H^{1/2}(\Gamma)} \big(\|h_{\mathcal{E}}^{1/2}\varrho\|_{L^2(\Gamma)} + \|g_2 - \tilde{g}_2\|_{H^{-1/2}(\Gamma)}\big) \\ &+ \|h_{\mathcal{E}}^{1/2} J_\tau\|_{L^2(\cup\mathcal{E})} \|h_{\mathcal{E}}^{-1/2}(\beta - B)\|_{L^2(\cup\mathcal{E})} \\ &+ \|h_{\mathcal{T}} \operatorname{curl}_{\mathcal{T}} P\|_{L^2(\Omega)} \|h_{\mathcal{T}}^{-1}(\beta - B)\|_{L^2(\Omega)} \\ &+ \|t_0 - \tilde{t}_0\|_{H^{-1/2}(\Gamma)} \|\mathcal{V}\partial\beta/\partial s\|_{H^{1/2}(\Gamma)} \Big\}. \end{aligned} \quad (4.25)$$

From the mapping properties of the single-layer potential operator and (4.8), we deduce

$$\left\|\mathcal{V}\frac{\partial\beta}{\partial s}\right\|_{H^{1/2}(\Gamma)} \leqslant c_8 \|d \cdot n - \nabla\alpha \cdot n\|_{H^{-1/2}(\Gamma)} \leqslant c_9 (\|d \cdot n\|_{H^{-1/2}(\Gamma)} +$$

$\|g\|_{H^{1/2}(\Gamma)} + \|\nabla\alpha\|_{L^2(\Omega)})$
$\leqslant c_{10}(\|d \cdot n\|_{H^{-1/2}(\Gamma)} + \|\delta\|_{H^{1/2}(\Gamma)} + \|\nabla\alpha\|_{L^2(\Omega)})$.(4.26)From the well-established properties of the Clément approximation, we quote from Carstensen (1997c), Carstensen & Verfürth (1997), Clément (1975) and Verfürth (1996) the estimate

$$\|h_{\mathcal{T}}^{-1}(\beta - B)\|_{L^2(\Omega)} + \|h_{\mathcal{E}}^{-1/2}(\beta - B)\|_{L^2(\cup\mathcal{E})} \leqslant c_{11}\|\nabla\beta\|_{L^2(\Omega)}. \tag{4.27}$$

Note also that $\|\nabla\beta\|_{L^2(\Omega)} = \|\mathrm{Curl}\,\beta\|_{L^2(\Omega)} \leqslant \|\nabla\alpha\|_{L^2(\Omega)} + \|d\|_{L^2(\Omega)}$. Hence and because of the positive definiteness of the single-layer potential and hypersingular integral operator, we can absorb the terms $\|d \cdot n\|_{H^{-1/2}(\Gamma)}$, $\|\delta\|_{H^{1/2}(\Gamma)}$, $\|\nabla\alpha\|_{L^2(\Omega)}$, and $\|d\|_{L^2(\Omega)}$ after employing (4.26) and (4.27) in (4.25). This concludes the proof of (4.3).

In the second part of the proof, we study the displacement error. For $e := u - U \in L^2(\Omega)$ there exists a unique $\eta \in H^1(\Omega)/\mathbb{R}$ with

$$\Delta\eta = e \text{ in } \Omega \quad \text{and} \quad \partial\eta/\partial n = 0 \text{ on } \Gamma. \tag{4.28}$$

Assuming an H^2-regular domain Ω we have $\nabla\eta \in H^1(\Omega)$ and the *a priori* estimates

$$\|\eta\|_{H^2(\Omega)} \leqslant c_{12}\|e\|_{L^2(\Omega)}. \tag{4.29}$$

In particular, we can utilize the Fortin operator, $s > 2$,

$$\Pi : H(\mathrm{div};\Omega) \cap L^s(\Omega)^2 \to \mathcal{M} \tag{4.30}$$

which satisfies the error estimate

$$\|h_{\mathcal{T}}^{-1}(\nabla\eta - \Pi\nabla\eta)\|_{L^2(\Omega)} \leqslant c_{13}\|D^2\eta\|_{L^2(\Omega)}, \tag{4.31}$$

a commuting diagram property

$$(\mathrm{div}(\nabla\eta - \Pi\nabla\eta), V) = 0 \quad \text{for all } V \in \mathcal{L}, \tag{4.32}$$

and is defined along the edges $\mathcal{E}$ to fulfil

$$\int_E V(\nabla\eta - \Pi\nabla\eta) \cdot n_E \,\mathrm{d}s = 0 \quad \text{for all } V \in \mathcal{L} \text{ and all } E \in \mathcal{E}. \tag{4.33}$$

We refer to Brezzi & Fortin (1991) for details and proofs.

According to (4.28), an integration by parts, and (4.32) we obtain

$$\|e\|^2_{L^2(\Omega)} = (u, \Delta\eta) - (U, \Delta\eta) = -(p, \nabla\eta) - (U, \mathrm{div}\,\Pi\nabla\eta). \tag{4.34}$$

Since $\Pi\nabla\eta \in \mathcal{M}$, (3.4) and (4.33) show

$$(U, \mathrm{div}\,\Pi\nabla\eta) = -(P, \Pi\nabla\eta). \tag{4.35}$$

(The boundary terms dissappear because of $\Pi\nabla\eta \cdot n = 0$ owing to $\nabla\eta \cdot n = 0$ and (4.33) for Raviart–Thomas and Brezzi–Douglas–Fortin–Marini elements.) From (4.32) and (4.33) we infer, for each $V \in \mathcal{L}$, with an elementwise integration by parts that

$$(\nabla_{\mathcal{T}} V, \nabla\eta - \Pi\nabla\eta) = 0. \tag{4.36}$$

Evaluating (4.35) and (4.36) in (4.34) we deduce (with another integration by parts involving d)

$$\begin{aligned}\|e\|^2_{L^2(\Omega)} &= (\nabla_{\mathcal{T}} V - P, \nabla\eta - \Pi\nabla\eta) - (d, \nabla\eta)\\ &= (\nabla_{\mathcal{T}} V - P, \nabla\eta - \Pi\nabla\eta) + (\operatorname{div} d, \eta) - \langle d\cdot n, \eta\rangle\\ &\leqslant c_{14}\,\|h_{\mathcal{T}}(P - \nabla_{\mathcal{T}} V)\|_{L^2(\Omega)}\|e\|_{L^2(\Omega)} + c_{15}\,\|d\cdot n\|_{H^{-1/2}(\Gamma)}\|e\|_{L^2(\Omega)}\\ &\quad + c_{16}\|h_{\mathcal{T}}(f + \operatorname{div} P)\|_{L^2(\Omega)}\|e\|_{L^2(\Omega)}. \end{aligned} \tag{4.37}$$

Here we used (4.14), (4.29) and (4.31). This concludes the proof of (4.4). □

Theorem 3 yields the *a posteriori* error estimate

$$\|p - P\|^2_{L^2(\Omega)} + \|(p - P)\cdot n\|^2_{H^{-1/2}(\Gamma)} + \|\xi - \Xi\|^2_{H_0^{1/2}(\Gamma)} \leqslant C\cdot\sum_{T\in\mathcal{T}}\eta(T)^2, \tag{4.38}$$

where (recall that $\mathcal{E}$ is the set of all edges)

$$\begin{aligned}\eta(T)^2 := h_T^2 &\int_T (|\operatorname{curl} P|^2 + |f + \operatorname{div} P|^2)\,\mathrm{d}x\\ &+ \sum_{E\in\mathcal{E}\wedge E\subset\partial T} h_E\big(\|J_\tau\|^2_{L^2(E)} + \|\mathcal{W}\Xi + (\mathcal{K}^* + 1)(P\cdot n - \tilde{t}_0)\|^2_{L^2(E)}\\ &+ \|t_0 - \tilde{t}_0\|^2_{L^2(E)}\big). \end{aligned} \tag{4.39}$$

This global reliable estimate is sharp in the sense that, up to higher-order approximation errors, the reverse inequality is true in a local form.

Let $\mathcal{N}(T)$ denote the union of all triangles that share (at least) one vertex with $T\in\mathcal{T}$.

THEOREM 4 Suppose P is a $\mathcal{T}$-piecewise polynomial and let f_T denote the integral mean of f on $T\in\mathcal{T}$. Then, there is an h_T-independent constant $c_{17} > 0$ (which depends only on c_θ and the piecewise polynomial degrees) such that for each $T\in\mathcal{T}$

$$\begin{aligned} c_{17}\,\eta(T)^2 &\leqslant \|p - P\|^2_{L^2(\mathcal{N}(T))}\\ &+ h_T^2\|(f - f_T)\|^2_{L^2(\mathcal{N}(T))} + h_T\|t_0 - \tilde{t}_0\|^2_{L^2(\Gamma\cap\partial T)}\\ &+ h_T\|W(\xi - \Xi)\|^2_{L^2(\Gamma\cap\partial T)} + h_T\|(K^* + 1)(p - P)\cdot n\|^2_{L^2(\Gamma\cap\partial T)}\\ &+ h_T\left\|\frac{\partial}{\partial s}V(p - P)\cdot n\right\|^2_{L^2(\Gamma\cap\partial T)} + h_T\left\|\frac{\partial}{\partial s}(K + 1)(\xi - \Xi)\right\|^2_{L^2(\Gamma\cap\partial T)}. \end{aligned} \tag{4.40}$$

Proof. As all the terms can be evaluated with inverse inequalities and approximation errors of higher order as indicated in Verfürth (1996), we may refer to Alonso (1996), Carstensen (1996a, b, 1997a) and Carstensen & Funken (1999b) and omit the details. □

REMARK 6 Summing (4.40) over all elements yields a global estimate in which the integral operator errors can be recast as in Carstensen (1996a) and Carstensen & Funken (1999b) adopting the arguments of Carstensen (1996b) for quasi-uniform meshes on the boundary. This shows

$$c_{18}\sum_{T\in\mathcal{T}}\eta(T)^2 \leqslant \|p-P\|^2_{L^2(\Omega)} \tag{4.41}$$
$$+\|h_{\mathcal{T}}(f-f_T)\|^2_{L^2(\Omega)}+\|h_{\mathcal{E}}^{1/2}(t_0-\tilde{t}_0)\|^2_{L^2(\Gamma)}$$
$$+h_{\Gamma,\max}/h_{\Gamma,\min}\cdot\left(\|\xi-\Xi\|^2_{H_0^{1/2}(\Gamma)}+h_{\Gamma,\max}\|\frac{\partial}{\partial s}(\xi-\tilde{\xi})\|^2_{L^2(\Gamma)}\right)$$
$$+h_{\Gamma,\max}/h_{\Gamma,\min}\cdot\left(\|(p-P)\cdot n\|^2_{H^{-1/2}(\Gamma)}+h_{\Gamma,\max}\|p\cdot n-\tilde{\phi}\|^2_{L^2(\Gamma)}\right).$$

Here, $h_{\Gamma,\max}$ (resp. $h_{\Gamma,\min}$) denotes the maximal (resp. minimal) mesh size of the boundary elements in $\mathcal{G}$ and $\tilde{\phi}$ denotes the $\mathcal{G}$-piecewise constant integral mean of $p\cdot n$, and the $\mathcal{G}$-piecewise affine $\tilde{\xi}$ approximates ξ in $H^1(\Gamma)$. This inequality establishes that the error indicator is generically efficient for triangulations with quasi-uniform meshes on the boundary. Indeed, for smooth data and solutions, the terms $\|h_{\mathcal{T}}(f-f_T)\|_{L^2(\Omega)}$, $\|h_{\mathcal{E}}^{1/2}(t_0-\tilde{t}_0)\|_{L^2(\Gamma)}$, $h_{\Gamma,\max}^{1/2}\|\frac{\partial}{\partial s}(\xi-\tilde{\xi})\|_{L^2(\Gamma)}$ and $h_{\Gamma,\max}^{1/2}\|p\cdot n-\tilde{\phi}\|_{L^2(\Gamma)}$ on the right-hand side of (4.41) are of higher order, $O(h_{\max}^{3/2})$. Hence, we generically obtain the reverse inequality for quasi-uniform meshes $\mathcal{G}$ on Γ, namely

$$c_{19}\sum_{T\in\mathcal{T}}\eta(T)^2 \leqslant \|p-P\|^2_{L^2(\Omega)}$$
$$+\|\xi-\Xi\|^2_{H_0^{1/2}(\Gamma)}+\|(p-P)\cdot n\|^2_{H^{-1/2}(\Gamma)}+O(h_{\max}^{3/2}) \tag{4.42}$$

(with an h-independent constant c_{19} that depends on $h_{\Gamma,\max}/h_{\Gamma,\min}$).

REMARK 7 The estimate (4.40) shows that $\eta(T)$ is a local estimator. Even for T at the interface Γ, the boundary contributions may be regarded as pseudo-local (according to the pseudo-locality of pseudo-differential operators).

5. An adaptive algorithm and its implementation

Given a local error indicator $\eta(T)$ which is (even locally) related to the local error (in Theorem 4), we may follow the standard approach in residual-based adaptive mesh-refining algorithms and employ the following scheme.

ALGORITHM 1

(a) Start with a coarse mesh $\mathcal{T}_k$, $k=0$.

(b) Solve the discrete problem for the actual mesh $\mathcal{T}_k$.

(c) Compute $\eta(T)$ for all $T\in\mathcal{T}_k$.

(d) Evaluate stopping criterion and decide whether to terminate or goto (e).

(e) Refine the element T (red refinement) provided

$$\tfrac{1}{2}\max_{T'\in\mathcal{T}_k}\eta(T')\leqslant\eta(T).$$

(f) Refine further elements (red-green-blue refinement) to avoid hanging nodes. Define resulting mesh as actual mesh $\mathcal{T}_k$, update k and goto (b).

REMARKS 8
(i) We refer to Verfürth (1996) for details on the red-green-blue refinement we employed.
(ii) The heuristic of Algorithm 1 is that a refinement of T with large $\eta(T)$ lowers the error near T. This is not strictly supported by Theorems 3 and 4 since the estimate (4.3) is global and the local estimate (4.39) includes nonlocal operators and so the error on the entire boundary Γ.

The adaptive algorithm is implemented using Matlab and we conclude this section with some remarks on the numerical Matlab realization before we report on numerical examples to illustrate the practical performance in the next section.

The dualities on the left-hand side, e.g. (P, Q), $(U, \operatorname{div} Q)$, $\langle \mathcal{V}P \cdot n, Q \cdot n\rangle$ and $\langle \mathcal{K}\Xi, Q \cdot n\rangle$ where P, Q, U, Ξ are piecewise constant or piecewise linear (scalar or vector valued) functions can be calculated almost analytically. On the right-hand side for given functions $f \in L^2(\Gamma)$, $u_0 \in H^1(\Gamma)$, and $t_0 \in L^2(\Gamma)$ we compute $\int_\Omega f\eta_j \,\mathrm{d}x$ via a mid-point quadrature rule on any triangle T and the integrals $\langle Q \cdot n, u_0\rangle$, $\langle t_0, V\rangle$ and the integral mean $\tilde{t}_0$ of t_0 are approximated by an eight-point Gaussian quadrature formula. (See Carstensen & Funken (1999a, b) and the literature quoted therein for terms with integral operators.)

In the first numerical example in the next section the potentials u and u_c and hence the gradient $p = \nabla u$ are known explicitly. Hence the $L^2(\Omega)$-norms of $u - U$ and $p - P$ can be calculated via the seven-point quadrature rule of order six from Abramowitz & Stegun (1984, formula 25.4.63c) on any triangle and the $H_0^{1/2}(\Gamma)$-norm of $\xi - \Xi$ (resp. $H^{-1/2}(\Gamma)$-norm of $(p-P)\cdot n$) by its equivalent quantity $\|\xi - \Xi\|_{\mathcal{W}}^2 := \langle \mathcal{W}(\xi - \Xi), \xi - \Xi\rangle$ (resp. $\|(p-P)\cdot n\|_{\mathcal{V}}^2 := \langle \mathcal{V}(p-P)\cdot n, (p-P)\cdot n\rangle$). This gives an approximation of the left-hand sides in (3.6) and (4.3).

The calculation of the integrals for the residuals in (4.38) over the finite element T and the boundary element Γ_k is performed as follows.

Since P is piecewise affine and since $f = 0$ in the numerical examples, the terms $\int_T |f + \operatorname{div} P|^2 \,\mathrm{d}x$, $\int_T |\operatorname{curl} P|^2 \,\mathrm{d}x$, and the jumps across the interior element boundaries in J_τ can be calculated exactly. The $L^2(\Gamma_k)$-norm of

$$2P|_T \cdot t_E - \frac{\partial}{\partial s}\left(2u_0 + (\mathcal{K}+1)\Xi - \mathcal{V}(P\cdot n - \tilde{t}_0)\right),$$

is approximated via a three-point Gaussian quadrature formula on each boundary element Γ_k. For $x_j \in \Gamma_k$ and $g \in C(\Gamma_k)$ the derivative $(\partial/\partial s)\, g(x_j)$ is replaced by its central difference operator $[g(x_{j+1}) - g(x_{j-1})]/|x_{j+1} - x_{j-1}|$ with a distance of nodes $|x_{j+1} - x_{j-1}| = |\Gamma_k|/20$. The terms

$$\int_{\Gamma_k} |\mathcal{W}\Xi + (\mathcal{K}^* + 1)(P\cdot n - \tilde{t}_0)|^2 \,\mathrm{d}s \qquad \text{and} \qquad \int_{\Gamma_k} |t_0 - \tilde{t}_0|^2 \,\mathrm{d}s$$

are also approximated with a three-point Gaussian quadrature rule.

6. Numerical examples

The following examples provide numerical evidence of the superiority of the adaptive mesh-refining Algorithm 1 in comparison with quasi-uniform mesh-refinement.

EXAMPLE 1 Let us consider the interface problem (1.1)–(1.5) on the L-shaped domain in Fig. 1 with exact solution

$$u(r,\theta) = r^{2/3}\sin(2\theta/3) \quad \text{and} \quad v(x,y) = \log(|(x+\tfrac{1}{2}, y-\tfrac{1}{2}|)$$

given in polar (resp. Cartesian coordinates) (r,θ) (resp. (x,y)).

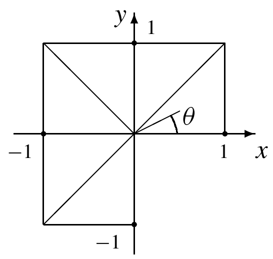

FIG. 1. L-shaped domain.

The solution has a typical corner singularity such that the convergence rate of the h-version with a uniform mesh does not lead to the optimal convergence rate even though the right-hand side is smooth.

Table 2 (resp. Table 3) displays the numerical results for a sequence of uniform meshes (resp. meshes generated by Algorithm 1). We show the number of degrees of freedom N, the energy-norm of the corresponding error

$$|||e_N||| := (\|p-P\|^2_{L^2(\Omega)} + \|(p-P)\cdot n\|^2_{\mathcal{V}} + \|\xi - \Xi\|^2_{\mathcal{W}})^{1/2},$$

the experimental convergence rate

$$\gamma_N := -\log(e_{N_j}/e_{N_{j+1}})/\log(\sqrt{N_j}/\sqrt{N_{j+1}})$$

of two subsequent meshes $\mathcal{T}_j$, $\mathcal{T}_{j+1}$, the estimated error η_N and the ratio $|||e_N|||/\eta_N$. The experimental convergence rate α, i.e. convergence as $O(h^\alpha)$ with a mesh size $h = O(N^{-1/2})$, is approximately $\alpha = 2/3$ for uniform meshes. The adaptive Algorithm 1 leads to a quasi-optimal linear convergence rate. The ratio $|||e_N|||/\eta_N$ has an upper bound ($\leqslant 0{\cdot}27$) in our numerical example which provides experimental evidence for the estimate (4.3).

EXAMPLE 2 As a more practical example, we consider $u_0 = 0$ and $t_0 = 0$ in (1.4) and (1.5). The unknown exact solution models the potential of a capacitor in an unbounded domain. The charge at boundaries $\Gamma_{D,1}$ and $\Gamma_{D,2}$ are ± 1, respectively. The geometry of Ω, Ω_c, Γ_C and Γ_D is depicted in Fig. 2, where the coarse grid is also shown.

TABLE 2

Errors $|||e_N|||$, convergence rates γ_N, error estimates η_N and ratio $|||e_N|||/\eta_N$ in Example 1 for a sequence of uniform meshes

N	$\|\|\|e_N\|\|\|$	γ_N	η_N	$\|\|\|e_N\|\|\|/\eta_N$
38	0·071278		1·1730	0·190
141	0·043649	0·724	0·5454	0·254
545	0·021398	0·652	0·3239	0·269
2145	0·011122	0·632	0·2117	0·264
8513	0·005977	0·628	0·1379	0·261
33921	0·003316	0·632	0·0887	0·261
135425	0·001895	0·636	0·0566	0·263

TABLE 3

Errors $|||e_N|||$, convergence rates γ_N, error estimates η_N and ratio $|||e_N|||/\eta_N$ in Example 1 for meshes generated by Algorithm 1

N	$\|\|\|e_N\|\|\|$	γ_N	η_N	$\|\|\|e_N\|\|\|/\eta_N$
38	0·2117324		1·1730	0·190
141	0·1316732	0·724	0·5454	0·254
463	0·0929464	0·584	0·3891	0·246
774	0·0664472	1·306	0·2832	0·244
1138	0·0513531	1·336	0·2359	0·228
1495	0·0414916	1·562	0·2034	0·214
2821	0·0286142	1·170	0·1376	0·220
3516	0·0247449	1·318	0·1255	0·211
6662	0·0178850	1·014	0·0936	0·203
9906	0·0143884	1·096	0·0751	0·205
15847	0·0112419	1·050	0·0594	0·202
25853	0·0087725	1·012	0·0469	0·200
40553	0·0069174	1·054	0·0370	0·201
64223	0·0054651	1·024	0·0295	0·199
100888	0·0043591	1·000	0·0236	0·198
160217	0·0034432	1·018	0·0187	0·198

Algorithm 1 produces a sequence of unstructured meshes as shown in Fig. 5. For the coarse mesh the problem behaves like a crack problem and as the mesh is increasingly refined around $\Gamma_{D,j}$, it models a domain with re-entrant corners of the Dirichlet boundary. The solution for this problem with $N = 51724$ (9th grid) is shown in Fig. 3 and a magnification of the adaptively refined mesh around $\Gamma_{D,1}$ is provided in Fig. 4. The meshes are highly refined at the corners of the Dirichlet boundary as expected. There is

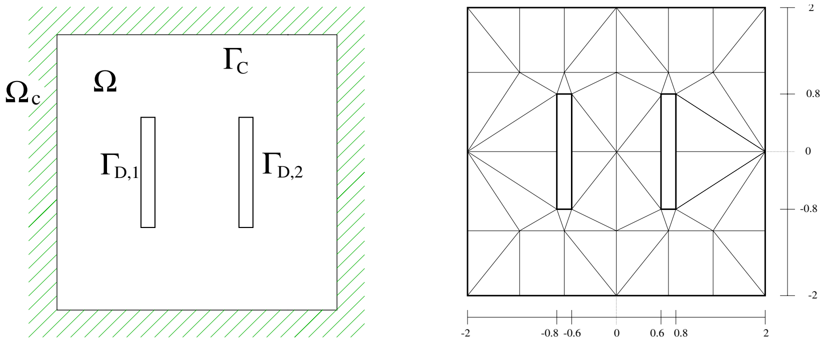

FIG. 2. Configuration of Example 2.

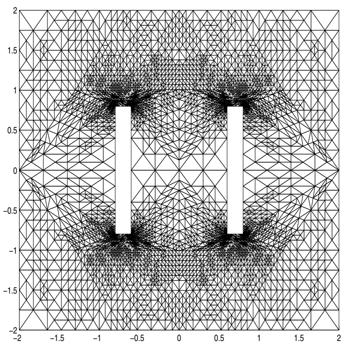

FIG. 3. Mesh $\mathcal{T}_9$ generated by Algorithm 1.

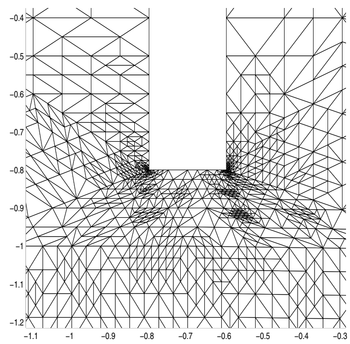

FIG. 4. Zoom view of the mesh near an inner corner of Fig. 3.

no additional refinement on the coupling boundary due to the coupling compared with pure FEM-modelling. As shown in Fig. 3 the refinement is symmetric around the x- and y-axes. The streamlines displayed provide knowledge about the gradients of the potential. Although we are using mixed finite elements in Ω the streamlines appear smooth, even near the coupling boundary.

In Fig. 6 we plot the *a posteriori* error estimate η_N for uniform and adaptive meshes (from Fig. 5). The convergence rate of η_N is approximately 1 for the adaptive meshes and 0·7 for uniform meshes. (A slope of $-\frac{1}{2}$ in 6 corresponds to an experimental convergence rate of 1 owing to $N \propto h^{-2}$ in two dimensions.) As expected, the *a posteriori* error estimate η_N decreases considerably faster for adaptively refined meshes with quasi-optimal convergence rate. This provides support for the adaptive mesh being more efficient than a uniform discretization.

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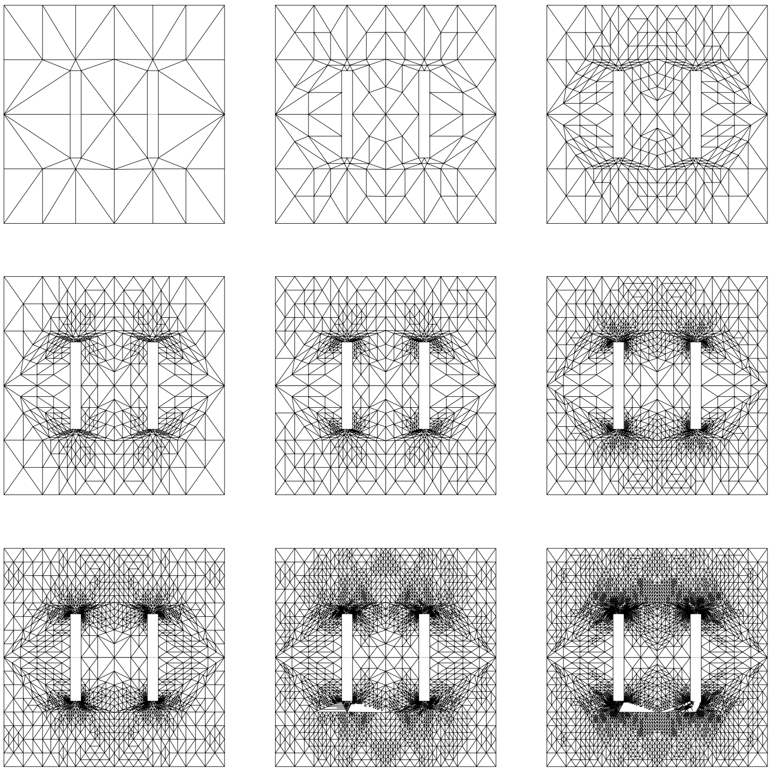

FIG. 5. Sequence of meshes $\mathcal{T}_k$, $(k = 1, \ldots, 9)$ generated by Algorithm 1.

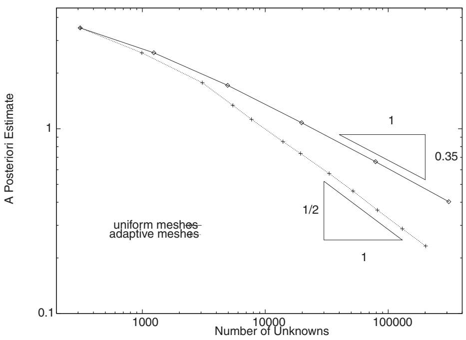

FIG. 6. Error estimates η_N for uniform and adapted meshes versus degrees of freedom N.

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