Calculating the energy-norm FEM-error for Reissner–Mindlin plates without known reference solution

C. Carstensen, K. Weinberg

Abstract The validation of (recently introduced conforming) finite element technologies for the numerical treatment of Reissner–Mindlin plate models requires comparisons with the unknown exact solution. Since mathematical results are often provided for the error in energy norms only it is not sufficient to compare a typical displacement or moment at one point of the domain. Instead of computing a reference solution on a very fine mesh and then providing a lot of data for the public we propose the storage of one (problem depending) constant \( C \) which then allows an error representation which merely involves known quantities. Based on this approach we could verify convergence rates which were theoretically predicted and give experimental evidence that new adaptive automatic mesh-refining algorithms yield superior approximations. Given any reasonable guess of \( C \) (computable from known quantities), our error representation yields an approximation for the unknown error. This establishes a method for a posteriori error control to be employed as a termination criterion.

1 Introduction

Mixed finite element schemes in general require deeper mathematical analysis as an arbitrary choice of ansatz and test spaces yields instabilities. Very recently, new finite element methods where introduced in the mathematical literature [1, 2, 3, 6, 8] for the effective numerical treatment of the Reissner–Mindlin model of the moderately thick plate

\[
\begin{align*}
\frac{t^2}{12} \Delta \theta + \frac{\mu}{6} \theta (\theta) + \mu (\nabla w - \theta) &= 0, \\
\mu \nabla (\nabla w - \theta) + f &= 0.
\end{align*}
\] (1.1)

The numerical verification of the theoretically predicted convergence rates for those schemes is either missing or problematic: The relation to the Kirchhoff plate model is employed in [6], but restricted to very thin plates where (1.1)–(1.2) is not designed for. The influence of the important parameter \( \alpha \) (cf. below) is studied in [2] only by comparing one typical displacement (whose convergence properties were not covered by the theory). The superiority of adaptive mesh-refining algorithms could not be shown experimentally in [10] because of the lack of a typical singular example with known reference solution.

The design of a benchmark with known exact solution to (1.1)–(1.2) is difficult: prescribing a general vertical displacement \( w \) and rotations \( \theta = (\theta_1, \theta_2) \) of the plate normal does not necessarily satisfy (1.1) (while we could choose \( f \) to satisfy (1.2)). Even more problematic is a typical choice of \((w, \theta)\) which reflects the finite element’s performance in practice.

Our work was motivated by a strange pre-asymptotic performance of recent finite elements in [10] which rather suggested errors in our code than the predicted linear convergence. Here, we establish an error representation for the energy norm \( \| \cdot \| \) denotes the norm in the Lebesgue space \( L^2(\Omega) \)

\[
e_h^2 := \| C^{1/2} (\theta - \theta_h) \|_2^2 + r^{-2} \| \theta - \theta_h - \nabla (w - w_h) \|_2^2 = C + \text{computable terms} \ (w_h, \theta_h). \tag{1.3}
\]

The constant \( C \) could be computed from \( f \) and the unknown \( w \). In practice we may use a finite element calculation on a very fine mesh to provide this constant \( C \) and then may use the second identity in (1.3) to calculate \( e_h \). Notice that we need to calculate \( C \) only once and thereafter everybody could use our error representation to calculate \( e_h \) on a different mesh with different conforming finite elements.

Given an approximation \( C_h \) to \( C \), obtained, e.g., by extrapolation techniques from computed moderately fine meshes, we can use (1.3) to calculate

\[
\eta_h^2 := C_h + \text{computable terms} \ (w_h, \theta_h) \tag{1.4}
\]

as an approximation to \( e_h \) which may serve as an a posteriori termination criterion. Numerical evidence for the efficiency of the error estimator \( \eta_h \) is provided.

The rest of the paper is organised as follows: Detailed notations, weak and discrete formulations are provided in Sect. 2. Our theoretical main result is stated and proven in Sect. 3. The first example of Sect. 4 compares our results with an analytical solution for the Kirchhoff equation on a square plate while the second of Sect. 5 illustrates the practical application of the strategy proposed. As a result

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we can establish theoretical predicted convergence rates and superiority of the adaptive algorithm from [5] over a uniform refinement by numerical experiments.

2 Weak formulation and finite element discretisation

The strong formulation (1.1)–(1.2) with the identity matrix I, the two-dimensional linear Green strain tensor $e(\vartheta) = \text{sym}(\nabla \vartheta)$ is recast with a scaled external load $g := f/(\mu k^2)$ and elasticity tensor $\mathcal{C}$,

$\mathcal{C}_e = \frac{1}{2} (12\mu k) I \text{ tr } e + e/(6k)$, with constant material coefficients $\mu, \lambda$ and constant plate thickness $t$. In its weak formulation the problem reads: Find $(w, \vartheta) \in H^1_0(\Omega) \times H^1_0(\Omega)$ such that, for all $(v, \varphi) \in H^1_0(\Omega) \times H^1_0(\Omega)^2$,

$$(e(\vartheta) ; \mathcal{C}e(\varphi)) + t^{-2} (\nabla w - \vartheta ; \nabla v - \varphi) = (g; w).$$

(2.1)

The Sobolev space $H^1_0(\Omega)$ contains all functions with a square-integrable gradient and homogeneous boundary conditions and $(; ;)$ denotes the $L^2(\Omega)$-scalar product:

$$(g; w) := \int_\Omega g \cdot w \, \text{d}x.$$ Because of the shear locking phenomena the finite element discretisation of (2.1) is not trivial. To rewrite (2.1) as a saddle point problem we follow Arnold and Brezzi [1] and introduce a shear variable

$$\gamma := (t^{-2} - \alpha)(\nabla w - \vartheta)$$

for a parameter $\alpha$ with $0 < \alpha = \alpha(x) < t^{-2}$. In this paper, the function $\alpha = \alpha(x)$ is a possibly discontinuous function which may vary with $x \in \Omega$ and may be different on different elements. We suppose that $0 < \alpha < t^{-2}$ is bounded away from $t^{-2}$ such that $(t^{-2} - \alpha)^{-1}$ is essentially bounded. With bilinear forms

$$a(\mathbf{w}, \mathbf{v}; \varphi, \varphi) := (e(\vartheta); \mathcal{C}e(\varphi)) + (\alpha(\nabla w - \vartheta); \nabla v - \varphi),$$

(2.3)

$$b(\mathbf{w}, \mathbf{v}; \eta) := (\nabla w - \vartheta; \eta),$$

(2.4)

$$c(\gamma; \eta) := ((t^2/(1-\alpha t^2))\gamma; \eta),$$

(2.5)

the continuous problem reads: Find $(w, \vartheta, \gamma) \in H^1_0(\Omega) \times H^1_0(\Omega)^2 \times L^2(\Omega)^2$ such that

$$a(\mathbf{w}, \mathbf{v}; \varphi, \varphi) + b(\mathbf{v}, \varphi, \gamma) = (g; \mathbf{v}),$$

(2.6)

$$b(\mathbf{w}, \vartheta; \eta) - c(\gamma; \eta) = 0$$

(2.7)

for all $(v, \varphi, \eta) \in H^1_0(\Omega) \times H^1_0(\Omega)^2 \times L^2(\Omega)^2$.

The conforming discretisation is described by replacing the continuous spaces $H^1_0(\Omega) \times H^1_0(\Omega)^2 \times L^2(\Omega)^2$ by subspaces $\mathcal{V}_w \times \mathcal{V}_\vartheta \times \mathcal{V}_\gamma \subset H^1_0(\Omega) \times H^1_0(\Omega)^2 \times L^2(\Omega)^2$ which are based on a mesh $\mathcal{T}$ (we neglect the subindex $h$ for simplicity).

Suppose $\mathcal{T}$ to be a regular triangulation in the sense of Ciarlet [7], i.e., $\mathcal{T}$ is a finite partition of $\Omega$ into closed triangles $T_1, T_2, \ldots, T_n$. Let $\mathcal{P}_k(\mathcal{T})$ be the linear space of $\mathcal{T}$-piecewise polynomials of degree $\leq k$,

$$\mathcal{P}_k(\mathcal{T}) := \{ u \in L^2(\Omega) : \forall T \in \mathcal{T}, u|_T \in \mathcal{P}_k(T) \},$$

(2.8)

and let $\mathcal{B}_3(\mathcal{T})$ be the space of $\mathcal{T}$-piecewise cubic bubble functions,

$$\mathcal{B}_3(\mathcal{F}) := \{ u \in C(\Omega) : \forall T \in \mathcal{T}, u|_T \in \mathcal{P}_3 \}.$$ and $u = 0$ on $\partial T$. (2.9)

Then, after Arnold and Brezzi [1], the following finite element scheme with $\mathcal{T}$-piecewise quadratic polynomials for the transverse displacement, $\mathcal{T}$-piecewise affines plus bubble modes for the rotation component, and $\mathcal{T}$-piecewise constants for the shear variable components,

$$\mathcal{H}_w \times \mathcal{H}_\vartheta \times \mathcal{L}_\gamma := (\mathcal{P}_2(\mathcal{T}) \cap H^1_0(\Omega))(\mathcal{P}_1(\mathcal{T}) \oplus \mathcal{B}_3(\mathcal{T})) \cap H^1_0(\Omega),$$

(2.10)

yields a stable discrete problem which reads: Find $(w_h, \vartheta_h, \gamma_h) \in \mathcal{H}_w \times \mathcal{H}_\vartheta \times \mathcal{L}_\gamma$ such that

$$a(w_h, \vartheta_h; v_h, \varphi_h) + b(v_h, \varphi_h; \gamma_h) = (g; v_h),$$

(2.11)

$$b(w_h, \vartheta_h; \eta_h) - c(\gamma_h; \eta_h) = 0$$

(2.12)

for all $(v_h, \varphi_h, \eta_h) \in \mathcal{H}_w \times \mathcal{H}_\vartheta \times \mathcal{L}_\gamma$.

3 Error representation

The error representation (1.3) is established below where $(w, \vartheta) \in H^1_0(\Omega) \times H^1_0(\Omega)^2$ denotes the exact solution to (1.1)–(1.2) and $(w_h, \vartheta_h) \in \mathcal{H}_w \times \mathcal{H}_\vartheta$ solves (2.11)–(2.12). Then, the right-hand side of (3.1) specifies what is called “computable terms $(w_h, \vartheta_h)$” in the introduction.

Theorem 1. With the real number

$$C := (g; w) = \|\mathcal{C}^{1/2}e(\vartheta)\|^2 + t^{-2} \|\vartheta - \nabla w\|^2$$

$$= \|\mathcal{C}^{1/2}e(\vartheta - \vartheta_h)\|^2 + t^{-2} \|\vartheta - \vartheta_h - \nabla(w - w_h)\|^2$$

$$= C - \|\mathcal{C}^{1/2}e(\vartheta_h)\|^2 - 2t^{-2} \|\nabla w_h - \vartheta_h\|^2$$

$$+ 2((t^{-2} - \alpha) \nabla w_h - \vartheta_h - \gamma_h; \nabla w_h - \vartheta_h),$$

(3.1)

Remark 1. The theorem holds for mixed boundary conditions as well with the only modification that, then,

$$C := \|\mathcal{C}^{1/2}e(\vartheta)\|^2 + t^{-2} \|\vartheta - \nabla w\|^2$$

equals the total exterior energy of the exact solution (which includes $(g; w)$ and other energy contributions on the boundary). The theorem holds for other conforming finite element schemes, for all inhomogeneous parameters $(x, \alpha)$, and arbitrary meshes as well with the same $(t$-depending) constant $C$.

Proof. Abbreviate the discretisation errors $e_w := w - w_h$ and $e_\vartheta := \vartheta - \vartheta_h$ and calculate

$$\|\mathcal{C}^{1/2}e(\vartheta)\|^2 + t^{-2} \|\nabla e_w - e_\vartheta\|^2$$

$$= (e(\vartheta); \mathcal{C}e(\vartheta + \vartheta_h)) - 2(e(\vartheta); \mathcal{C}e(\vartheta_h))$$

$$+ t^{-2}(e_\vartheta - \nabla e_w; \vartheta + \vartheta_h - \nabla w - \nabla w_h)$$

$$- 2t^{-2}(e_\vartheta - \nabla e_w; \vartheta_h - \nabla w_h)$$

$$= \|\mathcal{C}^{1/2}e(\vartheta)\|^2 + t^{-2} \|\vartheta - \nabla w\|^2$$

$$+ \|\mathcal{C}^{1/2}e(\vartheta_h)\|^2 - 2t^{-2} \|\vartheta_h - \nabla w_h\|^2$$

$$- 2(e(\vartheta); \mathcal{C}e(\vartheta_h)) - 2t^{-2}(\vartheta - \nabla w; \vartheta_h - \nabla w_h).$$

(3.2)
The weak form (2.7) implies (2.2) and the difference of (2.6) and (2.11) yields
\[
(\varepsilon(e_0); \mathbf{C} \varepsilon(\partial_h)) = - (\gamma - \gamma_h + \alpha(\nabla_w - e_0); \nabla w_h - \partial_h) .
\]  
(3.3)

Then, direct calculations show that (3.2) equals C plus
\[
- \| \mathbf{C}^{1/2} \varepsilon(\partial_h) \|_2^2 - t^{-2} \| \partial_h - \nabla w_h \|_2^2
+ 2((t^{-2} - \alpha)(\nabla w_h - \partial_h) - \gamma; \nabla w_h - \partial_h) .
\]  
(3.4)

The representation of C as the exterior work follows with \( v = w \) and \( \varphi = \theta \) in (2.6)–(2.7).

Based on the identity (3.1), we proposed the following procedures on the computation of \( e_0 \) resp. \( \eta_h \).

(i) For a sequence \( (\mathcal{F}_k) \) of uniformly refined meshes compute the discrete displacement \( (w_{hk}) \) and calculate
\( C_{hk} := (g; w_{hk}) \). Compute C by extrapolation of the sequence \( (C_{hk}) \). (Alternatively, compute C from an analytical solution or with a numerical reference solution on a very fine mesh.) Step (i) has to be performed once for an example and is determined by \( \Omega, g \), and \( t \).

(ii) For any (known) discrete solution \( (w_h, \partial_h) \) from any conforming finite element scheme on any mesh, compute
\[
C = \| \mathbf{C}^{1/2} \varepsilon(\partial_h) \|_2^2 - t^{-2} \| \nabla w_h - \partial_h \|_2^2
+ 2((t^{-2} - \alpha)(\nabla w_h - \partial_h) - \gamma; \nabla w_h - \partial_h)
\]  
(3.5)

which then is (an approximation up to computer precision and errors in C to) \( e_0^2 \).

Subsequent tests can take C from the literature and merely implement step (ii).

(iii) Steps (i)–(ii) can be linked to compute \( \eta_h \) for a posteriori error control in actual computations on at minimal three consecutive meshes, e.g., within an adaptive mesh-refining algorithm: For each sequence of three consecutive successively refined meshes \( \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \) compute the corresponding discrete solutions \( (w_1, \partial_1, \gamma_1), (w_2, \partial_2, \gamma_2), (w_3, \partial_3, \gamma_3) \) and the parameters \( c_0 := (g; w_1), c_2 := (g; w_2), c_3 := (g; w_3) \). With Aitken-\( \Delta^2 \)-extrapolation method [9] consider \( C_k := c_k - (c_k - c_{k-1})^2/(c_{k-1} - 2c_{k-2} + c_{k-1}) \) (or \( C_{hk} \) provided by any other extrapolation scheme) as an improved approximation to \( C = (w; g) \). Given \( C_{hk} \) and \( (w_h, \partial_h) := (w_3, \partial_3) \) employ (1.4) to compute \( \eta_h \) as a known approximation to the (unknown) energy error \( e_0 \) of \( (w_h, \partial_h) \).

### 4 Example for comparison with Kirchhoff solution

Since the solutions of (1.1)–(1.2) tend (in a very weak topology) to the solutions of Kirchhoff’s equations when \( t \to 0 \), we compare our computation of C with the exact value \( (g; w) \) for the (different) solution \( w \) defined, for \( (x, y) \in \Omega := (-1/2, +1/2)^2 \), by
\[
w(x, y) = (x - 1/2)^2(x + 1/2)^2(y - 1/2)^2(y + 1/2)^2 .
\]  
(4.1)

and with \( f = t^2/12(\lambda + 2\mu)\Delta w, \mu = 4.2, \lambda = 3.6 \) and a thickness of \( t = 0.001 \). Given (4.1) we obtained \( (g; w) = 2.332361 \cdot 10^{-10} \) as an approximation to the unknown C in (1.3) for the original Eqs. (1.1)–(1.2).

Using symmetry we calculated only one quarter \([0, 0.5]^2\) of the domain \( \Omega \). The first mesh \( \mathcal{F}_1 \) consisted of 4 squares each divided into 2 triangles with \( N = 52 \) degrees of freedom. Table 1 displays the results from uniform mesh refinements and \( \alpha = 100 \) where the errors \( e_k \) were computed with (1.3) for \( C = 2.3324 \cdot 10^{-10} \) in every refinement step \( k \), \( e_k := e_k \). The last two columns of Table 1 show the two error terms computed analytically with \( w \) from (4.1). As the last column has to be multiplied by \( t^{-2} = 10^6 \) and added to the second last, the digits of the last column agree very nicely with \( e_k^2 \) computed as proposed in this paper.

In this example, the reference solution provided an accurate approximation for the constant C. To illustrate the quality of the Aitken-\( \Delta^2 \)-method [9], we computed the entries in the third column of Table 1 called \( (c_1, \ldots, c_k) \) by \( c_k := (g; w_k) \) where \( w_k \) is (part of) the discrete solution on the mesh \( \mathcal{F}_k \). The transformation
\[
C_k = c_{k-2} - \frac{(c_{k-1} - c_{k-2})^2}{c_{k-1} - 2c_{k-2} + c_{k-1}}
\]  
(4.2)

provided the numbers \( (C_3, \ldots, C_k) \) displayed in the third column in Table 1. One important experimental observation of this note is that the extrapolated quantities are apparently very accurate approximations to the exact value C and this yields a good a posteriori error estimation by \( \eta_h \) computed as described in (iii) of the previous section.

Given that \( c_1, \ldots, c_k \) are known (with \( k \geq 3 \)) we replaced \( C_k \) in (1.4) by the extrapolated value (4.2) to compute \( \eta_h := \eta_h \) shown in the sixth column of Table 1.

To study convergence rates of the finite element scheme (2.10) for different values of \( \alpha \), the relative error \( e_N := e_0/\sqrt{C} \) is computed as before and plotted in Fig. 1. Here and below the error \( e_N \) is plotted versus the number of degrees of freedom \( N \) in a log/log-scale; whence a slope \(-1/2 \) in the figures below corresponds to an experimental convergence rate 1 owing to \( N \propto h^{-2} \) in two dimensions. For comparisons, triangles with slopes 1 and 3/2 are shown in the figure as well.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( N )</th>
<th>( c_k \times 10^{-10} )</th>
<th>( e_k \times 10^{-5} )</th>
<th>( C_k \times 10^{-10} )</th>
<th>( \eta_k \times 10^{-5} )</th>
<th>( | \mathbf{C}^{1/2} \varepsilon(e_0) |_2^2 \times 10^{-16} )</th>
<th>( | e_0 - \nabla w_h |_2^2 \times 10^{-16} )</th>
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<td>2.3323</td>
<td>0.77930</td>
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</tr>
<tr>
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<td>57216</td>
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<td>0.38257</td>
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<td>0.38259</td>
<td>0.14765</td>
<td>0.14641</td>
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We observe in Fig. 1 that $a$ influences the relative error dramatically. The results for $a = 1$ yield the theoretically predicted convergence rate $1$, while the convergence for $a = 100$ and $a = 10^4$ partially are improved. We remark that the slopes for $a = 100$ are nearly $3/2$ for the first four meshes and $1$ for the last two. Theoretical investigations suggest an improved rate of $3/2$ if $a = O(h)$ [2].

The experimental convergence rate for $a = 10^4$ is approximately $2$ while in the pre-asymptotic range (e.g., the first entry) we even observe a non-monotone (whence non-systematic) error behaviour. Improved convergence results for smaller $a$ were theoretically predicted in [6].

To assess the quality of the a posteriori error estimator $\eta_h$, as described in (ii) of Sect. 4, the relative error estimators $\eta_N := \eta_k / \sqrt{C_k}$ are plotted in Fig. 1 (for $k = 3, \ldots, 6$). The entries for $e_k$ and $\eta_k$ lay on top of each other which is numerical evidence for a high accuracy of the proposed error estimation.

5 Numerical example for practical error computation
Consider the L-shaped plate $(-1, 1)^2 \setminus [0, 1]^2$ of thickness $t = 0.01$. The (unknown) exact solution is expected to be singular near the origin at the re-entering corner even though the load is uniformly distributed and the material parameters are constant (with the values from Sect. 4). We start our finite element computation with a coarse mesh shown in Fig. 2 and refine uniformly but presumably sub-optimally.

Table 2 displays the sequence of obtained and extrapolated values for the constants $c_k$ and $C_k$. The energy error $e_k$ of the finite element approximation on the mesh $\mathcal{T}_k$ were obtained with (1.3) and the extrapolated value $C_5 = 4.0609 \times 10^{-3}$.

Table 2. Numerical results in the Example of Sect. 5 for uniform mesh-refinements and $a = 100$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>$c_k \times 10^{-9}$</th>
<th>$e_k \times 10^{-5}$</th>
<th>$C_k \times 10^{-9}$</th>
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</table>
Table 3. Numerical results in the Example of Sect. 5 for adapted mesh-refinements and $x = 100$

<table>
<thead>
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<th>$k$</th>
<th>$N$</th>
<th>$c_k \times 10^{-9}$</th>
<th>$e_k \times 10^{-5}$</th>
<th>$C_k \times 10^{-9}$</th>
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</table>

The last two columns display the error $\eta_k$ calculated with $c_k$ as explained in (iii) of Sect. 3. The convergence of the extrapolated values $C_k$ is very fast and the resulting error estimation is very good: $\eta_k$ is a very good estimate of $e_k$.

To assess the quality of the extrapolation for a non-uniform mesh, we employed an adaptive mesh-refining algorithm from [5]. The initial mesh $\mathcal{T}_0$ and the mesh $\mathcal{T}_9$ after 9 refinement are shown in Fig. 2. The numerical values are provided in Table 3 and obtained as corresponding values in Table 2.

A comparison of the extrapolated values $C_k$ shows as reasonable convergence but not necessarily to the same values. As the uniform meshes may inherit asymptotic error expansions form symmetry of meshes, we regard the uniform calculation (which involves much finer meshes) as the more accurate approximation (although we have no proof for that) and computed $e_k$ in Table 3 with the same value $C = 4.0609 \times 10^{-5}$ as in Table 2.

We observe again an accurate estimation of the energy error $e_k$ by $\eta_k$ (the values for $e_k$ obtained with a slightly different value such as $3.7580 \times 10^{-5}$ were almost the same).

To assess the improvement of adaptive mesh-refinements over a sequence of uniform meshes, Fig. 3 displays $e_N = e_k / \sqrt{C}$ and $\eta_N := \eta_k / \sqrt{C_k}$ versus the degrees of freedom $N$. The adaptive refining algorithms leads to a fast reduction of the error compared to the uniform refinements. This is the first numerical evidence that adaptive algorithms are superior for the Reissner–Mindlin plate and so indeed use-full tools.

References