A posteriori error estimators for convection–diffusion eigenvalue problems

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A B S T R A C T

A posteriori error estimators for convection–diffusion eigenvalue model problems are discussed in Heuveline and Rannacher (2001) [17] in the context of the dual-weighted residual method (DWR). This paper directly addresses the variational formulation rather than the non-linear ansatz of Becker and Rannacher for some convection–diffusion model problem and presents a posteriori error estimators for the eigenvalue error based on averaging techniques. Two different postprocessing techniques attached to the DWR paradigm plus two new dual-weighted a posteriori error estimators are also presented. The first new estimator utilises an auxiliary Raviart–Thomas mixed finite element method and the second exploits an averaging technique in combination with ideas of DWR. The six a posteriori error estimators are compared in three numerical examples and illustrate reliability and efficiency and the dependence of generic constants on the size of the eigenvalue or the convection coefficient.

1. Introduction

While the numerical approximation of eigenvalues of symmetric second-order elliptic partial differential equations (PDEs) with real eigenpairs is relatively well understood, much less is known about non-symmetric problems with possibly complex eigenvalues. A posteriori error estimators for symmetric eigenvalue problems can be found in [13, 20, 22, 23, 31]. The convergence of the adaptive finite element method (AFEM) for the symmetric case is considered in [9, 14, 15, 28]. A posteriori error estimators for some non-symmetric eigenvalue problems can be found in [11, 17, 18]. It is the aim of this paper to review the results of Heuveline and Rannacher in a direct approach rather than in the non-linear setting of the DWR paradigm following [1, 2, 17]. These results are also applicable to the averaging techniques as for the symmetric eigenvalue problem in [22]. Numerical experiments indicate that the efficiency indices for the residual-type a posteriori error estimators depend strongly on the convection coefficient ρ. Therefore, this paper investigates the dual-weighted residual paradigm from Becker and Rannacher [2, 3, 1] and presents two new dual-weighted a posteriori error estimators. The first new estimator is based on the Raviart–Thomas mixed finite element method (MFEM) [6, 27] of first-order and the second one on averaging techniques. Hence, they are named dual-weighted mixed (DWM) and dual-weighted averaging (DWA) a posteriori estimators. The paper

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presents numerical evidence that the DWR methodology in combination with the \( L^2 \) interpolation scheme of [32] is empirical reliable and efficient for unstructured triangular meshes while [17] is restricted to structured meshes because of the approximation of the weights by second-order difference quotients.

The convection–diffusion model eigenvalue problem reads: Seek an eigenpair \((\lambda, u) \in C \times \{ H^1_0(\Omega; C) \cap H^1_{lin}(\Omega; C) \} \) with

\[
- \Delta u + \beta \cdot \nabla u = \lambda u \quad \text{in } \Omega.
\]  

(1.1)

The given data \( \beta \in H(div; \Omega; \mathbb{R}^2) \) is assumed to be divergence free in the bounded Lipschitz domain \( \Omega \subset \mathbb{R}^2 \), i.e., \( \int_\Omega \nabla \varphi \cdot \beta \, dx = 0 \) for all \( \varphi \in V := H^1_0(\Omega; C) \).

The weak problem considers the two complex Hilbert spaces \( V \) with energy norm \( \| \cdot \| = \| \cdot \|_{H^1(\Omega; C)} \) which is a norm on \( V \) and \( W := L^2(\Omega; C) \) with norm \( \| \cdot \|_{L^2(\Omega; C)} \). The weak form reads: Seek an eigenpair \((\lambda, u) \in C \times V \) with \( \| u \| = 1 \) such that

\[
a(u, v) = \lambda b(u, v) \quad \text{for all } v \in V.
\]

(1.2)

The bilinear form \( a(\cdot, \cdot) \) is elliptic and continuous in \( V \) and the bilinear form \( b(\cdot, \cdot) \) is continuous, symmetric and positive definite, and hence induces a norm \( \| \cdot \| := b(\cdot, \cdot)^{1/2} \) on \( W \). For the above model problem, \( \| \cdot \| \equiv \| \cdot \|_{L^2(\Omega; C)} \) and the bilinear forms (where \( \overline{\cdot} \) denotes complex conjugation) read

\[
a(u, v) = \int_\Omega (\nabla u \cdot \nabla \overline{v} + (\beta \cdot \nabla u) \overline{v}) \, dx \quad \text{and} \quad b(u, v) = \int_\Omega u \overline{v} \, dx.
\]

Since \( \beta \) is assumed to be divergence free, an integration by parts yields

\[
\int_\Omega (\beta \cdot \nabla \overline{v}) v \, dx = - \int_\Omega (\beta \cdot \nabla \overline{v}) v \, dx.
\]

Hence, for all \( v \in V \), it holds that

\[
| \| v \| |^2 = \text{Re}(a(v, v)).
\]

Thus, the ellipticity constant (which is one) of the bilinear form \( a(\cdot, \cdot) \) is independent of \( \beta \).

The analysis of the non-symmetric eigenvalue problem requires the dual eigenvalue problem: Seek a (dual) eigenpair \((\lambda^*, u^*) \in C \times V \) with \( \| u^* \| = 1 \) such that

\[
a(v, u^*) = \lambda^* b(v, u^*) \quad \text{for all } v \in V.
\]

Since the embedding of \( V \) in \( W \) is continuous and compact, the spectral theory for compact operators [19,24] is applicable. The Riesz-Schauder theorem shows that the primal and dual spectra consist of finite or countable infinite many eigenvalues with no finite accumulation point. In particular, the algebraic multiplicities are finite.

Throughout this paper, suppose that \( \lambda \) is a simple eigenvalue in the sense that the algebraic multiplicity and hence the geometric multiplicity is one and that \( \lambda \) is well separated from the remaining part of the spectrum.

Given any finite-dimensional subspace \( V_0 \subset V \), the discrete problems read: Seek primal and dual (discrete) eigenpairs \((\lambda_i, u_i) \) and \((\lambda_i^*, u_i^*) \) with \( \| u_i \| = 1 = || u_i^* || \) such that

\[
a(u_i, v_i) = \lambda_i b(u_i, v_i) \quad \text{for all } v_i \in V_i \quad \text{and} \quad a(v_i, u_i^*) = \lambda_i^* b(v_i, u_i^*) \quad \text{for all } v_i \in V_i.
\]

(1.3)

The primal and dual eigenvalues \( \lambda_j \) and \( \lambda_j^* \) as well as the primal and dual discrete eigenvalues \( \lambda_{ij} \) and \( \lambda_{ij}^* \) are connected by

\[
\lambda_j = \lambda_{ij} \quad \text{for } j = 1, 2, 3, \ldots \quad \text{and} \quad \lambda_{ij}^* = \lambda_{ij}^* \quad \text{for } f = 1, \ldots, \text{dim}(V_i).
\]

The abstract a priori theory yields the maximal upper bounds in terms of the maximal mesh-size \( H \),

\[
| \lambda - \lambda_j | \leq H^2, \quad \| u - u_j \| \leq H^2, \quad \| u^* - u^*_j \| \leq H^2,
\]

where \( 0 < s_1 \leq 1 \) and \( 0 < s_2 \leq 1 \) depend on the regularity of the primal and dual eigenfunctions [[24], Chapter 10.3]. This paper employs standard notation on Lebesgue and Sobolev spaces and norms. Moreover, \( x \leq y \) denotes an estimate \( x \leq Cy \) with some generic constant \( C > 0 \), which is independent of the maximal mesh-size \( H \). Similarly \( x \approx y \) abbreviates the inequalities \( x \leq y \) and \( y \leq x \).

The outline of the remaining parts of this paper is as follows. In Section 2 an optimal error estimate for the eigenvalue error is derived. For this, the basic algebraic properties and identities of the non-symmetric eigenvalue problem are reviewed. In contrast to [17], the direct variational formulation is used, rather than the more general non-linear DWR framework of Becker and Rannacher [1,2]. The weak regularity assumptions and the suboptimal \( L^2 \) error estimate of [17] prove the \( L^2 \) contribution to the residual identity to be of higher-order. Section 3 summarises some old and some new results on several a posteriori error estimators, namely the residual, the averaging, and the dual-weighted DWR1, DWR2, DWM and DWA a posteriori error estimators. Section 4 describes the adaptive finite element method, the interpolation scheme, used for the calculation of the weights, and the computation of the error estimators. In Section 5 the error estimators are compared in numerical benchmarks on three different domains for higher eigenvalues and various convection coefficients. Section 6 draws some conclusions.
2. Algebraic properties

This section is devoted with the primal and dual residual and the estimation of the eigenvalue and energy error in the primal and dual eigenfunctions.

For the primal and dual discrete eigenpairs $(\lambda_i, u_i)$ and $(\tilde{\lambda}_i, u'_i)$, the residuals are defined

$$\text{Res}_i := a(u_i, \cdot) - \lambda_i b(u_i, \cdot) \in V^* \quad \text{and} \quad \text{Res}'_i := a(\cdot, u'_i) - \tilde{\lambda}_i b(\cdot, u'_i) \in V'$$

for the dual space $V'$. Notice that $V_i \subseteq \ker(\text{Res}_i)$ and $V_i \subseteq \ker(\text{Res}'_i)$.

It is the goal of this section to derive the following optimal error estimate for the eigenvalue error of simple eigenvalues

$$|\tilde{\lambda}_i - \lambda_i| \lesssim ||\text{Res}_i||^2 + ||\text{Res}'_i||^2$$

(2.1)

which is valid only for $H_i \ll 1$. Throughout this paper let $e_i := u - u_i$ and $e'_i := u' - u'_i$.

**Lemma 2.1** (Primal–dual error residual identity). Suppose that $(\lambda_i, u_i)$ and $(\tilde{\lambda}_i, u'_i)$ are the discrete primal and dual eigenpairs to the primal and dual eigenpairs $(\lambda, u)$ and $(\tilde{\lambda}, u')$. Then it holds that

$$(\lambda - \lambda_i)(b(u, u') + b(u_i, u'_i) - b(e_i, e'_i)) = \text{Res}_i(e'_i) + \text{Res}'_i(e_i).$$

**Proof.** Direct algebraic manipulations and the definition of the residuals and using that $\lambda = \overline{\tilde{\lambda}}, \lambda_i = \overline{\tilde{\lambda}_i}$ leads to

$$a(u_i, u' - u'_i) - \lambda_i b(u_i, u' - u'_i) + a(u - u_i, u'_i) - \overline{\lambda_i}b(a - u_i, u'_i) = a(u_i, u' - u'_i) - \lambda_i b(u_i, u'_i) + a(u, u'_i) - \overline{\lambda_i}b(u, u'_i)$$

$$= (\lambda - \lambda_i)b(u, u') + (\overline{\lambda} - \overline{\lambda_i})b(u'_i).$$

**Lemma 2.2.** Suppose that the maximal mesh-size $H_i$ tends to zero as $\ell \to \infty$, then

$$\lim_{\ell \to \infty} b(e_i, e'_i) = 0 \quad \text{and} \quad \lim_{\ell \to \infty} b(u_i, u'_i) = b(u, u').$$

**Proof.** The convergence of $||e_i||$ and $||e'_i||$ implies the convergence of $||e_i||$ and $||e'_i||$ to zero as $\ell \to \infty$ because of the compact embedding. Hence, the assertions follow from $|b(e_i, e'_i)| \leq ||e_i||||e'_i||$ and

$$|b(u_i, u'_i) - b(u_i, u'_i)| = |b(u - u_i, u') + b(u_i, u' - u'_i)| \leq ||e_i|| + ||e'_i||.$$

**Remark 2.1.** Since all eigenvalues converge as $H_i \to 0$, $\lambda_i$ is, as $\lambda$, a simple eigenvalue for sufficiently small $H_i$. For a vector $z \in \mathbb{R}^m$ let $z^T$ denotes its complex conjugate transposed vector. The condition number $1 / |y^T B x_i|$ of the discrete eigenvalue $\lambda_i$ is defined for right and left eigenvectors $x_i$ and $y_i$ of the algebraic eigenvalue problems

$$A_i x_i = \lambda_i B x_i \quad \text{and} \quad y_i^T A_i = \overline{\lambda_i} y_i^T B_i,$$

with non-symmetric convection–diffusion matrix $A_i$ and symmetric positive definite mass matrix $B_i$. [16, Section 7.2.2]. It is known that $y_i^T B x_i \neq 0$ for simple eigenvalues and that $|y_i^T B x_i| \gg 0$ if the simple eigenvalue is well separated from the remaining part of the spectrum. Hence, for well separated simple eigenvalues considered in this paper, it is reasonable to assume $b(u, u') \neq 0$. Furthermore, $1 / |b(u, u')|$ is the condition number of the continuous eigenvalue $\lambda$ and

$$|b(u, u') + b(u_i, u'_i) - b(e_i, e'_i)| \to 2 |b(u, u')|$$

as $H_i \to 0$.

Suppose that $\lambda$ is simple such that $b(u, u') \neq 0$ and let $\ell \gg 1$ be such that the maximal mesh-size $H_i$ of the triangulation $T_\ell$ is sufficiently small, i.e.,

$$\max\{|e_i||, ||e'_i||| < \min\{1, |b(u, u')| / 2\}. \tag{2.2}$$

Then $|b(u, u')| < |b(u, u') + b(u_i, u'_i) - b(e_i, e'_i)| < 3$, where the lower bound follows from

$$|b(u, u') + b(u_i, u'_i) - b(e_i, e'_i)| = |2b(u, u') - b(u, u' - u'_i) - b(u - u_i, u')|$$

$$\geq 2 |b(u, u')| - |b(u, u' - u'_i) + b(u - u_i, u')|$$

$$\geq 2 |b(u, u')| - ||e_i|| - ||e'_i|| ||u' - u'_i||$$

$$= 2 |b(u, u')| - ||e'_i|| - ||e_i||.$$
and (2.2). Thus for simple eigenvalues $\lambda$ it holds that

$$|\lambda - \lambda_\ell| \approx |\text{Res}_\ell(e_\ell) + \text{Res}'_\ell(e_\ell)|. \quad (2.3)$$

This implies the suboptimal eigenvalue error estimate

$$|\lambda - \lambda_\ell| \leq ||\text{Res}_\ell||_\ell^2 + ||\text{Res}'_\ell||_\ell^2. \quad (2.4)$$

**Remark 2.2.** The proof of the following Lemma 2.3 applies a suboptimal $L^2$ error estimate that is based on the weak regularity assumption of the eigenvalue $\lambda$ with the eigenspace $E(\lambda)$. That is a condition on

$$a_\ell(\cdot, \cdot) = a(\cdot, \cdot) - \lambda b(\cdot, \cdot),$$

on the quotient space $V/E(\lambda)$ in the sense that

$$|||w||| \leq C \sup_{v \in V/E(\lambda)} \frac{|a_\ell(v, w)|}{||v||} \quad \text{for all} \quad w \in V/E(\lambda).$$

The constant $C$ depends on the distance of $\lambda$ to all other distinct eigenvalues and does not depend on the mesh-size. This weak regularity assumption implies the suboptimal $L^2$ error estimates [17, (70)–(71)]

$$||e_\ell|| \leq ||\text{Res}_\ell||_\ell^2 + |\lambda - \lambda_\ell| \quad \text{and} \quad ||e'_\ell|| \leq ||\text{Res}'_\ell||_\ell^2 + |\lambda - \lambda_\ell|. \quad (2.5)$$

**Lemma 2.3 (Energy estimate).** Suppose that $b(u, u') \neq 0$, the maximal mesh-size $H$ is sufficiently small according to (2.2), and $(\lambda_\ell, u_\ell)$ and $(\lambda'_\ell, u'_\ell)$ are the discrete primal and discrete dual eigenpairs to the primal and dual eigenpairs $(\lambda, u)$ and $(\lambda', u')$. Then it holds that

$$||e_\ell|| + ||e'_\ell|| \leq ||\text{Res}_\ell||_\ell^2 + ||\text{Res}'_\ell||_\ell^2.$$

**Proof.** Since $b(u, u) = 1 - b(u_\ell, u_\ell)$, the eigenvalue Eqs. (1.2) and (1.3) imply that

$$a(e_\ell, e_\ell) = \lambda_\ell + \lambda_\ell - \lambda b(u_\ell, u_\ell) - a(u_\ell, u_\ell).$$

The relation

$$\lambda_\ell b(u_\ell, u_\ell) = \lambda_\ell b(u_\ell, u_\ell) = \lambda_\ell \text{Re} b(u, u_\ell) - i\lambda_\ell \text{Im} b(u, u_\ell)$$

leads to

$$a(e_\ell, e_\ell) = (\lambda + \lambda_\ell)(1 - \text{Re} b(u_\ell, u_\ell)) + i(\lambda_\ell - \lambda) \text{Im} b(u_\ell, u_\ell) + \lambda_\ell b(u_\ell, u_\ell) - a(u_\ell, u_\ell).$$

From $0 = \text{Im} ||u_\ell||^2 - \text{Im} b(u_\ell, u_\ell)$ it follows that

$$a(e_\ell, e_\ell) = (\lambda + \lambda_\ell)(1 - \text{Re} b(u_\ell, u_\ell)) + i(\lambda_\ell - \lambda) \text{Im} b(u - u_\ell, u_\ell) + \lambda_\ell b(u_\ell, u_\ell) - a(u_\ell, u_\ell).$$

Since

$$2\text{Re} b(u, u_\ell) = ||u||^2 + ||u_\ell||^2 - ||e_\ell||^2 = 2 - ||e_\ell||^2,$$

this implies

$$||e_\ell||^2 = \text{Re} a(e_\ell, e_\ell) \leq ||\text{Res}_\ell(e_\ell)|| + |\lambda + \lambda_\ell||e_\ell|| + \frac{|\lambda + \lambda_\ell|}{2} ||e_\ell||^2. \quad (2.6)$$

The suboptimal estimates (2.4) and (2.5) imply

$$|\lambda - \lambda_\ell| + ||e_\ell|| \leq ||\text{Res}_\ell||_\ell^2 + ||\text{Res}'_\ell||_\ell^2. \quad (2.7)$$

Since $||\cdot|| \leq ||\cdot||_\ell$, the inequalities (2.6), (2.7) yield

$$||e_\ell|| \leq ||\text{Res}_\ell||_\ell^2 + ||\text{Res}'_\ell||_\ell^2.$$ \hspace{1cm} \Box

**Theorem 2.4 (Eigenvalue Error Estimate).** Suppose that $b(u, u') \neq 0$, the maximal mesh-size $H$ is sufficiently small such that (2.2) holds and let $(\lambda_\ell, u_\ell)$ and $(\lambda'_\ell, u'_\ell)$ be the discrete primal and discrete dual eigenpairs to the primal and dual eigenpairs $(\lambda, u)$ and $(\lambda', u')$ for the simple eigenvalue $\lambda$. Then it holds that

$$|\lambda - \lambda_\ell| \leq ||\text{Res}_\ell||_\ell^2 + ||\text{Res}'_\ell||_\ell^2.$$


Proof. The aforementioned estimate (2.3), the Cauchy–Schwarz inequality and Lemma 2.3 lead to
\[ |\lambda - \lambda_0| \leq |\text{Res}_S(e^*_s)| + |\text{Res}_D(e^*_D)| \leq \|\text{Res}_S\|^2 + \|\text{Res}_D\|^2. \]
\[ \square \]

3. A posteriori error estimates

This section is devoted to the residual, averaging and dual-weighted residual a posteriori error estimators for the eigenvalue error of simple eigenvalues. The first two residual and averaging based a posteriori error estimators make use of Theorem 2.4
\[ |\lambda - \lambda_0| \leq \|\text{Res}_S\|^2 + \|\text{Res}_D\|^2. \]
Here, the dual norms of the primal and dual residuals are bounded separately. The DWR based a posteriori error estimators are derived from the asymptotic estimate (2.3) for simple eigenvalues,
\[ |\lambda - \lambda_0| \approx |\text{Res}_S(e^*_S) + \text{Res}_D(e^*_D)|, \]
where the constant tends to \(1/(2|b(u, u^*|)\) as \(H \rightarrow 0\). In general the dual-weighted error estimators avoid any additional inequality, such as approximation properties, with unknown constants. Thus, they are robust with respect to strong convection which is also confirmed by the numerical examples in Section 5. One question that arises from the computation of \(\text{Res}_D(e^*_S)\) or \(\text{Res}_D(e^*_D)\) is the calculation of the unknown errors \(e^*_S\) and \(e^*_D\). The rather heuristic approach of [11] states that it is numerically reliable and efficient to approximate these quantities which occur only in the weights. The idea is that one does not need to approximate the weights with higher accuracy than the size of the residual terms. In practice, the unknown primal and dual solutions \(u, u^*\) are replaced by solutions of a higher-order method or by higher-order interpolation. In Section 4 a higher-order interpolation ansatz for general triangular meshes is described which leads to numerically reliable and efficient dual-weighted a posteriori error estimators.

Throughout this paper, suppose \((T_i)\), is a family of shape-regular triangulations of \(\Omega\) into triangles, i.e. each \(T \in T_i\) is a closed triangle, \(\bar{\Omega} = \bigcup_{T \in T_i} T\), for any two distinct triangles \(T_1, T_2 \in T_i\) and \(T_1 \cap T_2\) is either empty, a common vertex or a common side. Suppose that the minimal angle of every triangle is uniformly bounded from below. The conforming finite element space of order \(k \in \mathbb{N}\) for the triangulation \(T_i\) is defined by
\[ P_k(T_i) := \{ v \in H^1(\Omega; \mathbb{C}) : \forall T \in T_i, v_T \text{ is polynomial of degree } \leq k \}. \]
Let \(V_i := P_k(T_i) \cap V\), and \(h_i := h_0(T_i)\) be such that \(h_0(T) := \text{diam}(T)\) for all \(T \in T_i\). Given a triangulation \(T_i\), define \(E_i\) as the set of inner edges and \(N_i\) as the set of inner nodes. Let \(h_T := \text{diam}(T)\) for \(T \in T_i\), and \(h_E := \text{diam}(E)\) for \(E \in E_i\). The jump of the discrete gradient \(\nabla u_e \in P_0(T_i)\) in normal direction \(v_k\) along an inner edge \(\partial T_+ \cap \partial T_- = E \in E_i\), for \(T_+, T_- \in T_i\), is denoted by \(|\nabla u_e| \cdot v_k\), \(v_k = v_k_{l+} \cdot v_k_{l-} \cdot v_k\) and \(\|\nabla u_e\| \cdot v_k = 0\) for boundary edges \(E \subset \partial \Omega\).

3.1. Residual estimator

The first a posteriori error estimator is the residual error estimator from [17].

Lemma 3.1. Let \((\lambda_i, u_i)\) and \((\lambda_i^*, u_i^*)\) be the discrete primal and discrete dual eigenpairs to the primal and dual eigenpairs \((\lambda, u)\) and \((\lambda^*, u^*)\). Then it holds that
\[ \|\text{Res}_S\|^2 \leq \sum_{T \in T_i} h_T^2 \beta \cdot \|\nabla u_i - \lambda_i u_i\|^2_{L^2(T)} + \sum_{E \in E_i} h_E \|\|\nabla u_i||\cdot v_E\|^2_{L^2(E)}, \]
\[ \|\text{Res}_D\|^2 \leq \sum_{T \in T_i} h_T^2 \|\cdot \nabla u_i^* - \lambda_i^* u_i^*\|^2_{L^2(T)} + \sum_{E \in E_i} h_E \|\|\nabla u_i^*\|\cdot v_E\|^2_{L^2(E)}. \]

Proof. Let \(v_i\) denote the Scott-Zhang interpolation of \(v\) onto \(V_i\). Then it holds that
\[ \text{Res}_S(v) = \text{Res}_S(v - v_i) = a(u_i, v - v_i) - \lambda_i b(u_i, v - v_i) \]
\[ = \sum_{T \in T_i, l \in T} \nabla u_i \cdot \nabla (v - v_i) + (\beta \cdot \nabla u_i)(v - v_i) dx - \lambda_i \int_T u_i (v - v_i) dx \]
\[ = \sum_{T \in T_i} \int_T (\beta \cdot \nabla u_i - \lambda_i u_i)(v - v_i) dx + \sum_{E \in E_i} \int_E \nabla u_i \cdot v_E (v - v_i) ds. \]

The approximation properties of the interpolation operator [29]
\[ \sum_{T \in T_i} \|h_T^{-1} (v - v_i)\|^2_{L^2(T)} + \sum_{E \in E_i} \|h_E^{-1/2} (v - v_i)\|^2_{L^2(E)} \leq \|v\|^2. \]
(3.1)
and the Cauchy–Schwarz inequality yield
\[
\text{Res}_1(v) = \sum_{T \in \mathcal{T}_h} h_T \left\| \beta \cdot \nabla u - \lambda e u \right\|_{L^2(T)}^{2} \left\| h_T^{-1} (v - u) \right\|_{L^2(T)}^{2} + \sum_{E \in \mathcal{E}_h} h_E^{1/2} \left\| \nabla u \right\|_{L^2(E)} \left\| h_E^{-1/2} (v - u) \right\|_{L^2(E)}^{2}
\]

For the second assertion notice that the dual bilinear form \( a'(u', v) \) reads in the model problem
\[
a'(u', v) = a(v, u') = \int_{\Omega} (\nabla v \cdot \nabla u' + (\beta \cdot \nabla v) \nabla u') \, dx.
\]

An integration by parts leads to
\[
a'(u', v) = \int_{\Omega} (\nabla u' \cdot \nabla v - (\beta \cdot \nabla v) u') \, dx \quad \text{for all} \ v \in V.
\]

The same arguments as for the first assertion lead to the assertion for \(||\text{Res}_1'||||\). \( \square \)

### 3.2. Averaging estimator

The averaging technique concerns operators
\[
A : \mathcal{P}_0(T) \to \{ V_{h}^2 \cap \mathcal{C}(\Omega)^2 \},
\]
with the model example
\[
A(\nabla u) := \sum_{z \in N'} \left( \int_{\Omega} \nabla u(z) \, dx \right) \varphi_z.
\]

Here and throughout this paper, \( \varphi_z \) denotes the nodal basis function for an inner node \( z \in N' \). Alternative averaging operators from [7] could be employed as well.

**Lemma 3.2.** Let \((\lambda, u)\) and \((\lambda', u')\) be the discrete primal and discrete dual eigenpairs to the primal and dual eigenpairs \((\lambda, u)\) and \((\lambda', u')\). Then it holds that
\[
|||\text{Res}_1||| \leq ||h_1(-d \nabla A(\nabla u)) + \beta \cdot \nabla u - \lambda e u|||_{L^2(\Omega)} + ||A(\nabla u) - \nabla u|||_{L^2(\Omega)},
\]
\[
|||\text{Res}_1'||| \leq ||h_1(-d \nabla A(\nabla u)) - \beta \cdot \nabla u' + \lambda e u'|||_{L^2(\Omega)} + ||A(\nabla u') - \nabla u'|||_{L^2(\Omega)}.
\]

**Proof.** As in the previous lemma, let \( v \) denote the Scott–Zhang interpolation of \( u \) onto \( V_h \), since \( A(\nabla u) \) is globally continuous the divergence theorem can be applied. This yields
\[
\text{Res}_1(v) = \text{Res}_1(v - v_t) = a(u_t, v - v_t) - \lambda_1 b(u_t, v - v_t)
\]
\[
= \int_{\Omega} (\nabla u_t - A(\nabla u_t)) \cdot (\nabla v - \nabla v_t) \, dx - \int_{\Omega} d \nabla A(\nabla u_t) (\nabla v - \nabla v_t) \, dx + \int_{\Omega} (\beta \cdot \nabla u_t - \lambda e u_t) (\nabla v - \nabla v_t) \, dx.
\]

Hölder's inequality leads to
\[
\text{Res}_1(v) \leq \sum_{T \in \mathcal{T}_h} \left( h_T^2 ||d \nabla A(\nabla u_t)) + \beta \cdot \nabla u_t - \lambda e u_t|||_{L^2(T)}^{2} \left\| h_T^{-1} (v - u) \right\|_{L^2(T)}^{2} \right)
\]
\[
+ \sum_{T \in \mathcal{T}_h} ||\nabla u_t - A(\nabla u_t)||_{L^2(T)} ||\nabla (v - u)||_{L^2(T)}^{2}
\]

Using the stability and the approximation property (3.1)
\[
\sum_{T \in \mathcal{T}_h} ||\nabla v_t||_{L^2(T)}^{2} \leq ||v||^{2} \quad \text{and} \quad \sum_{T \in \mathcal{T}_h} ||h_T^{-1} (v - u)\||_{L^2(T)}^{2} \leq ||v||^{2},
\]

together with the Cauchy–Schwarz inequality yield
\[
\text{Res}_1(v) \leq \left( ||h_1(-d \nabla A(\nabla u_t)) + \beta \cdot \nabla u_t - \lambda e u_t|||_{L^2(\Omega)} + ||A(\nabla u_t) - \nabla u_t|||_{L^2(\Omega)} \right) ||v||.
\]

In the same way one proves the assertion for \(|||\text{Res}_1'||||\). \( \square \)

### 3.3. DWR1 estimator

The first DWR a posteriori error estimator (DWR1) is derived from the DWR ansatz as in [17] or [1] plus a result from [8].
Lemma 3.3. Let the eigenfunctions \( u, u^* \in H^2(\Omega) \cap H^1(T) \), \( H^1(T) \) denote the broken space of piecewise \( H^1 \) Sobolev functions, \((\lambda, u)\) and \((\lambda^*, u^*)\) be the discrete primal and discrete dual eigenpairs to the primal and dual eigenpairs \((\lambda, u)\) and \((\lambda^*, u^*)\), and

\[
\eta_T := \| \beta \cdot \nabla u - \lambda_i u \|_{L^2(T)} + h_T^{-1/2} \| \nabla u \|_{L^p(\partial T)}, \\
\eta^*_T := -\beta \cdot \nabla u^* - \lambda^*_i u^* \|_{L^2(T)} + h_T^{-1/2} \| \nabla u^* \|_{L^p(\partial T)}. \tag{3.2}
\]

Then it holds that

\[
| \text{Res}_i(e_i^*) | + | \text{Res}_i'(e_i) | \leq \sum_{T \in \mathcal{T}} h_T^2 \eta_T \| \nabla u^* \|_{L^2(\Omega_T)} + \sum_{T \in \mathcal{T}} h_T^2 \eta^*_T \| \nabla e_i \|_{L^2(\partial T)} + \text{HOT}
\]

for suitable fixed subsets \( \Omega_T \subseteq \Omega \), which contain \( T \in \mathcal{T}_e \), with skeleton \( \cup \mathcal{E}_\partial \), and a higher-order term

\[
\text{HOT} := \sum_{T \in \mathcal{T}} h_T^2 \eta_T \| \nabla u^* \|_{L^2(\Omega_T)} + \sum_{T \in \mathcal{T}} h_T^2 \eta^*_T \| \nabla e_i \|_{L^2(\partial_T)}.
\]

\[\text{Proof.} \quad \text{Suppose } u \in H^2(\Omega), \text{ then integration by parts and Hölder’s inequality show that}
\]

\[
\text{Res}_i(v) = \sum_{T \in \mathcal{T}} \int_T \nabla u_i \cdot (\beta - \lambda_i u_i)(\nabla v - \nabla v_i)dx \\
\leq \sum_{T \in \mathcal{T}} h_T^{-1/2} \| \nabla u_i \|_{L^2(\partial T)} \| \nabla v - \nabla v_i \|_{L^2(\Omega_T)} + \| \beta \cdot \nabla u_i - \lambda_i u_i \|_{L^2(\partial T)} \| v - v_i \|_{L^2(\Omega_T)} \leq \sum_{T \in \mathcal{T}} \eta_T \omega_T.
\]

Here, \( \eta_T \) is as defined in (3.2) and

\[
\omega_T := \| v - v_i \|_{L^2(\Omega_T)} + h_T^{1/2} \| v - v_i \|_{L^2(\partial T)}.
\]

Let \( v = I_T v \in V_T \) be the nodal interpolant of \( v \). The interpolation estimate [5]

\[
\| v - I_T v \|_{L^2(\Omega_T)} \leq h_T^4 \| D^2 v \|_{L^2(\Omega_T)}
\]

leads to

\[
\text{Res}_i(v) \leq \sum_{T \in \mathcal{T}} h_T^2 \eta_T \| D^2 v \|_{L^2(\Omega_T)}.
\]

In [17] \( D^2 v \) is locally approximated on each quadrilateral \( Q \) by \( D^2 v|_Q \) using finite differences. While this is an appropriate ansatz for structured meshes, for general triangular meshes considered here this is not suited. In [8] it is shown that \( v \in H^3(\Omega_T) \) implies

\[
\| D^2 v \|_{L^2(\Omega_T)} \leq c_1 h_T^{1/2} \| \nabla v \|_{L^2(\cup \mathcal{E}_{\Omega_T})} + c_2 \| \nabla v - v_i \|_{L^2(\partial T)}^{1/2}.
\]

The constant \( c_1 \) depends on the shape of elements and \( c_2 \) on \( \| v \|_{H^1(\partial T)} \). This leads to the estimate

\[
| \text{Res}_i(e_i) | \leq \sum_{T \in \mathcal{T}} h_T^{1/2} \eta_T \| \nabla u_i \|_{L^2(\cup \mathcal{E}_{\Omega_T})} + \text{HOT},
\]

with higher-order term

\[
\text{HOT} = \sum_{T \in \mathcal{T}} h_T^2 \eta_T \| \nabla e_i \|_{L^2(\partial T)}.
\]

Note that the jump term is formally equivalent to the energy norm and that HOT involves an extra factor of \( h_T^{1/2} \) compared to the other term of the estimate. Following the argumentation for the primal residual yields the assertion for the dual residual

\[
| \text{Res}_i'(e_i) | \leq \sum_{T \in \mathcal{T}} h_T^{1/2} \eta_T \| \nabla u_i \|_{L^2(\cup \mathcal{E}_{\Omega_T})} + \text{HOT},
\]

with the higher-order term

\[
\text{HOT} = \sum_{T \in \mathcal{T}} h_T^2 \eta_T \| \nabla e_i \|_{L^2(\partial T)}.
\]

Remark 3.1. From the theory in [8] it remains open to choose the fixed size of the patches \( \Omega_T \) containing \( T \in \mathcal{T} \). However, the numerical examples of Section 5 suggest, that, surprisingly, \( \Omega_T = T \) and thus \( \cup \mathcal{E}_{\Omega_T} = \partial T \) might be sufficient. This seems to be in agreement with [1].
3.4. DWR2 estimator

The second DWR estimator (DWR2) according to [1] reads as follows. Observe that this error estimator involves the unknown exact primal and dual errors \( e_i \) and \( \overline{e}^i \). In the numerical examples of Section 5, these errors will be approximated by the interpolation described in Section 4.

**Lemma 3.4.** The unknown exact errors \( e_i \) and \( \overline{e}^i \) satisfy

\[
| \text{Res}_i(e_i^r) + \text{Res}_i^*(e_i^r) | = \sum_{T \in \mathcal{T}_h} \int_T (\beta \cdot \nabla u_i - \lambda_i u_i) \overline{e}^i \, dx + \sum_{E \in \mathcal{E}_h} \int_E (\nabla u_i) \cdot \overline{e}^i, \quad \text{and} \quad e_i^r = \overline{e}^i
\]

\[
+ \sum_{T \in \mathcal{T}_h} \int_T (-\beta \cdot \nabla u_i^r - \lambda_i u_i^r) e_i \, dx + \sum_{E \in \mathcal{E}_h} \int_E (\nabla u_i^r) \cdot e_i, \quad \text{and} \quad e_i^r = \nabla e_i^r
\]

**Proof.** An integration by parts leads to

\[
\text{Res}_i(e_i^r) = a(u_i, u_i^r - u_i^r) - \lambda_i b(u_i, u_i^r) = \sum_{T \in \mathcal{T}_h} \int_T (\beta \cdot \nabla u_i - \lambda_i u_i)(u_i^r - u_i^r) \, dx + \sum_{E \in \mathcal{E}_h} \int_E (\nabla u_i) \cdot e_i, \quad \text{and} \quad e_i^r = \overline{e}^i
\]

Similarly,

\[
\text{Res}_i^*(e_i^r) = a(u - u_i, u_i^r - u_i^r) = \sum_{T \in \mathcal{T}_h} \int_T (-\beta \cdot \nabla u_i^r - \lambda_i u_i^r)(u_i^r - u_i^r) \, dx + \sum_{E \in \mathcal{E}_h} \int_E (\nabla u_i^r) \cdot e_i, \quad \text{and} \quad e_i^r = \nabla e_i^r
\]

\[\square\]

3.5. DWM estimator

Utilising the non standard Raviart–Thomas solution of an auxiliary problem leads to a new approach for a dual-weighted a posteriori error estimator. Note that this error estimator involves the unknown exact primal and dual errors \( e_i \) and \( \overline{e}^i \) as well as their unknown gradients \( \nabla e_i \) and \( \nabla \overline{e}^i \). In practice these errors need to be approximated as described in Section 4.

**Lemma 3.5.** Let the two mixed finite element functions \((q_M, u_M) \in \text{RT}_0(\mathcal{T}_h) \times \mathcal{P}_0(\mathcal{T}_h)\) and \((q_M^*, u_M^*) \in \text{RT}_0(\mathcal{T}_h) \times \mathcal{P}_0(\mathcal{T}_h)\) be the solutions of the equilibrium conditions

\[
- \text{div}(q_M) + \beta \cdot q_M = f_i \quad \text{in} \quad \Omega \quad \text{and} \quad q_M - \nabla u_M = 0 \quad \text{in} \quad \Omega,
\]

\[
- \text{div}(q_M^*) - \beta \cdot q_M^* = f_i^* \quad \text{in} \quad \Omega \quad \text{and} \quad q_M^* - \nabla u_M^* = 0 \quad \text{in} \quad \Omega,
\]

with right-hand sides \(f_i, f_i^* \in \mathcal{P}_0(\mathcal{T}_h)\) given by \(f_{i|T} := h_i^2 f_i \lambda_i u_i\) and \(f_{i|^*} := h_i^2 f_i^* \lambda_i u_i^*\) for \(T \in \mathcal{T}_h\). Then the unknown exact errors \( e_i \) and \( \overline{e}^i \) satisfy

\[
| \text{Res}_i(e_i^r) + \text{Res}_i^*(e_i^r) | \leq \int_\Omega (\nabla u_i - q_M) \cdot \nabla \overline{e}^i \, dx + \int_\Omega (\nabla u_i^r - q_M^*) \cdot \nabla e_i \, dx + \int_\Omega (\beta \cdot (\nabla u_i - q_M)) \overline{e}^i \, dx
\]

\[
- \int_\Omega (\beta \cdot (\nabla u_i^r - q_M^*)) e_i \, dx \quad \text{HOT}
\]

with the higher-order term

\[
\text{HOT} = \left| \int_\Omega (f_i - \lambda_i u_i) \overline{e}^i \, dx + \int_\Omega (f_i^* - \lambda_i u_i^*) e_i \, dx \right|
\]

**Proof.** By the definition of the auxiliary problem for \(q_M\) and integration by parts it holds that

\[
\text{Res}_i(e_i^r) = \int_\Omega \nabla u_i \cdot \nabla \overline{e}^i \, dx + \int_\Omega (\beta \cdot \nabla u_i - \lambda_i u_i) \overline{e}^i \, dx = \int_\Omega (\nabla u_i - q_M) \cdot \nabla \overline{e}^i \, dx + \int_\Omega \beta \cdot (\nabla u_i - q_M) \overline{e}^i \, dx + \int_\Omega (f_i - \lambda_i u_i) \overline{e}^i \, dx.
\]


\[
\int_\Omega (f_i - \lambda_i u_i) \overline{e}^i \, dx \leq \| f_i - \lambda_i u_i \| \| \overline{e}^i \| \leq \frac{1}{H} \left( \sum_{T \in \mathcal{T}_h} h_i^2 \| \lambda_i \nabla u_i \|_2^2 \right)^{1/2} \| \overline{e}^i \|.
\]

Note that \( \| \overline{e}^i \| \) is of the same convergence order as \( | \lambda - \lambda_i | \) and that the last term involves an additional term of order \( O(H) \). Therefore, this term is formally of higher-order compared to \( | \lambda - \lambda_i | \). The same argumentation leads to
\[
\text{Res}_i^*(e_i) = \int_{\Omega} \nabla \overline{u}_i^* \cdot \nabla e_i \, dx + \int_{\Omega} (-\beta \cdot \nabla \overline{u}_i^* - \lambda_i \overline{u}_i^*) e_i \, dx \\
= \int_{\Omega} (\nabla u_i^* - \overline{u}_i^*) \cdot \nabla e_i \, dx - \int_{\Omega} \beta \cdot (\nabla u_i^* - \overline{u}_i^*) e_i \, dx + \int_{\Omega} (f_i^* - \lambda_i \overline{u}_i^*) e_i \, dx.
\]

The last term is again a formally higher-order term. □

### 3.6. DWA estimator

The second new a posteriori error estimator makes use of the ideas of the DWR2 estimator. The new aspect proposed here is not to use integration by parts to obtain a residual term but to involve the averaged gradients \( A(\nabla u_i) \) and \( \overline{A(\nabla u_i)} \) and then to do integration by parts. Again this error estimator involves the unknown exact primal and dual errors \( e_i \) and \( \overline{e_i} \) which have to be approximated as described in Section 4.

**Lemma 3.6.** The unknown exact errors \( e_i \) and \( \overline{e_i} \) satisfy

\[
| \text{Res}_i(e_i') + \text{Res}_i^*(e_i) | = \left| \int_{\Omega} (\nabla u_i - A(\nabla u_i)) \cdot \nabla \overline{e}_i \, dx + \int_{\Omega} (\nabla \overline{u}_i - A(\nabla \overline{u}_i)) \cdot \nabla e_i \, dx \\
+ \int_{\Omega} (-d_i v(A(\nabla u_i)) + \beta \cdot \nabla u_i - \lambda_i u_i) \overline{e}_i \, dx + \int_{\Omega} (-d_i v(A(\nabla \overline{u}_i)) - \beta \cdot \nabla \overline{u}_i - \lambda_i \overline{u}_i) e_i \, dx \right|.
\]

**Proof.** An addition and subtraction of the averaging term \( A(\nabla u_i) \) and an integration by parts yields

\[
\text{Res}_i(e_i') = a(u_i, u_i' - u_i^*) - \lambda_i b(u_i, u_i' - u_i^*) = \int_{\Omega} (\nabla u_i - A(\nabla u_i)) \cdot \nabla \overline{e}_i \, dx + \int_{\Omega} (-d_i v(A(\nabla u_i)) + \beta \cdot \nabla u_i - \lambda_i u_i) \overline{e}_i \, dx.
\]

Analogously it follows

\[
\text{Res}_i^*(e_i) = a(u - u_i, u_i' - \overline{u}_i^*) - \overline{\lambda_i} b(u - u_i, u_i' - \overline{u}_i^*) = \int_{\Omega} (\nabla \overline{u}_i - A(\nabla \overline{u}_i)) \cdot \nabla e_i \, dx + \int_{\Omega} (-d_i v(A(\nabla \overline{u}_i)) - \beta \cdot \nabla \overline{u}_i - \lambda_i \overline{u}_i) e_i \, dx.
\]

□

### 4. Adaptive finite element method

The adaptive finite element method (AFEM) generates a sequence of meshes \( T_0, T_1, \ldots \) and associated discrete subspaces \( V_0 \subseteq V_1 \subseteq \ldots \subseteq V \) with discrete primal and discrete dual eigenpairs \((\lambda_i, u_i), (\lambda_i', u_i')\). A typical loop from \( V_i \) to \( V_{i+1} \) consists of the steps

**SOLVE → ESTIMATE → MARK → REFINE.**

#### 4.1. Solve

The primal and dual generalized algebraic eigenvalue problems

\[
A_i x_i = \lambda_i B_i x_i \quad \text{and} \quad y_i^k B_i = \overline{\lambda_i} y_i^k B_i
\]

are solved with an algebraic eigensolver. Here, the coefficient matrices are the non-symmetric convection–diffusion matrix \( A_i \) and the symmetric positive definite mass matrix \( B_i \). The right and left eigenvectors \( x_i \) and \( y_i \) represent the eigenfunctions

\[
u_i = \sum_{k=1}^{\dim(V_i)} x_i^k \varphi_k \quad \text{and} \quad u_i = \sum_{k=1}^{\dim(V_i)} y_i^k \varphi_k,
\]

with respect to the basis \( \langle \varphi_1, \ldots, \varphi_{\dim(V_i)} \rangle \) of \( V_i \).

#### 4.2. Estimate

Since the weight-terms \( e_i \) and \( \overline{e_i} \) in the dual-weighted a posteriori error estimators involve the unknown solutions \( u \) and \( \overline{u} \), they have to be approximated. In the following experiments those functions are approximated by averaging \( A(u_i) \in P_2(T_i) \) of \( u_i \in P_1(T_i) \) and \( \overline{A(u_i)} \in P_2(T_i) \) of \( \overline{u}_i \in P_1(T_i) \) on the mesh \( T_i \). In contrast to the recovery of a gradient as in [33], the \( L^2 \) recovery of [32] is used here which is similar but uses different interpolation points. The post-processing is based on element patches \( \omega_T := \cup_{T \subseteq \omega} \omega_T \) for \( T \subseteq T_i \), where \( \omega_T := \cup_{T \subseteq \omega} T \) is the nodal patch. The nodal and edge degrees of freedom for the interpolated \( P_2(T_i) \) function are computed for each element separately by a global least square quadratic
polynomial fitting. The interpolation points for the least square fitting are the nodal points of \( \omega_T \) as displayed in Fig. 1. After all local values are computed, a global \( P_2(\mathcal{T}_h) \) function is obtained by taking the arithmetic mean values for each node and midpoint of an edge of \( \mathcal{T}_h \).

In [4] an alternative way of computing the estimator \( \eta_{DWR2} \) based on nodal values is presented. The analysis of this error estimator makes use of a special interpolation operator. This operator assumes that the mesh \( \mathcal{T}_h \) results from uniform refinement of a coarser mesh and considers the nodal values as values for a higher-order \( P_2 \) basis on the coarser grid. The interpolation scheme presented here does not assume any structure of the mesh.

The step ESTIMATE of the AFEM loop involves an appropriate a posteriori error estimator. In the numerical examples of Section 5 the following error estimators are compared. Since the residual identity depends on the eigenvalue condition number the condition number needs to be approximated for efficient a posteriori error control with efficiency indices close to one. In Section 5 it is shown empirically that the approximation \( 1/(2b(u_i, u_i')) \) is efficient.

The first a posteriori error estimator is the residual estimator

\[
\eta_{i,R} = \frac{1}{2} \left| b(u_i, u_i') \right| \sum_{T \in \mathcal{T}_h} \left( h_T^2 \| \beta \cdot \nabla u_i - \lambda_i u_i \|^2_{L^2(T)} + \sum_{e \in \partial T} h_E \| \nabla u_i \cdot v_e \|^2_{L^2(\partial T)} \right)
\]

The second a posteriori error estimator is the averaging estimator

\[
\eta_{i,A} = \frac{1}{2} \left| b(u_i, u_i') \right| \sum_{T \in \mathcal{T}_h} \left( \| A(\nabla u_i) - \nabla u_i' \|^2_{L^2(T)} + h_T^2 \| -div(A(\nabla u_i)) + \beta \cdot \nabla u_i - \lambda_i u_i \|^2_{L^2(T)} \right)
\]

The third a posteriori error estimator is the DWR1 estimator where the higher-order terms are neglected

\[
\eta_{i,DWR1} = \frac{1}{2} \left| b(u_i, u_i') \right| \left( \sum_{T \in \mathcal{T}_h} h_T^{3/2} \| \nabla u_i' \cdot v_e \|^2_{L^2(\partial T)} + \sum_{T \in \mathcal{T}_h} h_T^{3/2} \| \nabla u_i \cdot v_e \|^2_{L^2(T)} \right),
\]

with \( \eta_T \) and \( \eta_f \) from (3.2).

The fourth a posteriori error estimator is the DWR2 estimator where the unknown solutions in the weights, \( u \) and \( u' \), are interpolated by \( A(u_T) \) and \( A(u_T') \) as described above

\[
\eta_{i,DWR2} = \frac{1}{2} \left| b(u_i, u_i') \right| \left( \sum_{T \in \mathcal{T}_h} \int_T (\| \nabla u_i' \cdot v_e \| (A(u_T') - A(u_T)) ds + \sum_{e \in \partial T} \int_{\partial T} (\| \nabla u_i' \cdot v_e \| (A(u_e) - u_e) ds 
\right)
\]

The local refinement indicators read

\[
\eta_T := \int_T (\beta \cdot \nabla u_i - \lambda_i u_i) (A(u_T') - A(u_T)) dx + \int_T (\| \nabla u_i' \cdot v_e \| (A(u_T') - A(u_T)) ds + \int_T (-\beta \cdot \nabla u_i' - \lambda_i u_i) (A(u_e) - u_e) ds 
\]

They are only necessary to determine the set of marked edges for refinement.

The fifth a posteriori error estimator utilised the auxiliary Raviart–Thomas mixed solutions \( q_m \) and \( q_m' \) and the averaged gradients \( A(\nabla u_i) \) and \( A(\nabla u_i') \).
Fig. 2. Refinement rules: sub-triangles with corresponding reference edges depicted with a second edge.

\[ \eta_{\text{DWM}} = \frac{1}{2} \left| \frac{1}{b(u_i, u'_i)} \right| \left[ \int_\Omega (\nabla u_i - q_M) \cdot (A(\nabla u_i) - \nabla u'_i) \, dx + \int_\Omega (\nabla u'_i - q_M) \cdot (A(\nabla u_i) - \nabla u_i) \, dx ight] + \int_\Omega \beta \cdot (\nabla u_i - q_M)(A(\nabla u_i) - \nabla u_i) \, dx - \int_\Omega \beta \cdot (\nabla u'_i - q_M)(A(u_i) - u_i) \, dx \right], \]

where the higher-order term is neglected. The local refinement indicators read

\[ \eta_T := \left| \int_T (\nabla u_i - q_M) \cdot (A(\nabla u_i) - \nabla u'_i) \, dx + \int_T (\nabla u'_i - q_M) \cdot (A(\nabla u_i) - \nabla u_i) \, dx + \int_T \beta \cdot (\nabla u_i - q_M)(A(\nabla u_i) - \nabla u_i) \, dx - \int_T \beta \cdot (\nabla u'_i - q_M)(A(u_i) - u_i) \, dx \right|. \]

The last error a posteriori error estimator uses both averaged gradients \( A(\nabla u_i) \) and \( A(\nabla u'_i) \) as well as interpolated \( L^2 \) functions \( A(u'_i) \) and \( A(\nabla u'_i) \) for the weights

\[ \mu_{\text{DNA}} = \frac{1}{2} \left| \frac{1}{b(u_i, u'_i)} \right| \left[ \int_\Omega (\nabla u_i - A(\nabla u_i)) \cdot (A(\nabla u_i) - \nabla u'_i) \, dx + \int_\Omega (\nabla u'_i - A(\nabla u'_i)) \cdot (A(\nabla u_i) - \nabla u_i) \, dx + \int_\Omega (-di v'(A(\nabla u_i))) \beta \cdot \nabla u_i - \lambda_i u_i) (A(\nabla u_i) - \nabla u_i) \, dx + \int_\Omega (-di v'(A(\nabla u'_i))) \beta \cdot \nabla u'_i - \lambda_i u'_i) (A(u_i) - u_i) \, dx \right]. \]

Here, the local refinement indicators read

\[ \eta_T := \left| \int_T (\nabla u_i - A(\nabla u_i)) \cdot (A(\nabla u_i) - \nabla u'_i) \, dx + \int_T (\nabla u'_i - A(\nabla u'_i)) \cdot (A(\nabla u_i) - \nabla u_i) \, dx + \int_T (-di v'(A(\nabla u_i))) \beta \cdot \nabla u_i - \lambda_i u_i) (A(\nabla u_i) - \nabla u_i) \, dx + \int_T (-di v'(A(\nabla u'_i))) \beta \cdot \nabla u'_i - \lambda_i u'_i) (A(u_i) - u_i) \, dx \right|. \]

4.3. Mark

Based on the refinement indicators, the set of elements \( M \subseteq T \) that are refined is specified in the algorithm Mark. Let \( M \) be the set of minimal cardinality for which the bulk criterion \([12]\),

\[ \theta \sum_{T \in T_i} \eta_T^2 \leq \sum_{T \in M_i} \eta_T^2 \]

is satisfied for a given bulk parameter \( 0 < \theta \leq 1 \).

4.4. Refine

Given the set \( M \subseteq T \) of marked elements, mark all edges of elements in \( M \) for refinement. The closure algorithm computes a superset of refined edges such that once an edge of a triangle is marked for refinement its reference edge is marked as well. The refinement \( T_{i+1} \) is obtained by application of the refinement rules from Fig. 2.

5. Numerical experiments

This section is devoted to numerical experiments and the empirical evidence of reliability, efficiency and stability for higher eigenvalues and strong convection coefficients. The numerical experiments on the unit square investigate the validity of the residual identity of Lemma 2.1 and the efficiency of the proposed eigenvalue condition number approximation. The experiments of the L shaped domain investigate the stability of the a posteriori error estimators for higher eigenvalues and the experiments on the slit domain their robustness in \( \beta \).

5.1. Unit square

As first example consider the convection–diffusion eigenvalue model problem (1.1) on the unit square \( \Omega = (0, 1) \times (0, 1) \). For constant convection coefficient \( \beta \), the exact eigenvalue with smallest real part reads \( \lambda = |\beta|^2/4 + 2\pi^2 \) \([26]\). The corresponding primal and dual eigenfunctions read
Two discrete primal and dual solutions are displayed in Fig. 3. To investigate the stability of the residual equation of Lemma 2.1 which depends on the condition number of the eigenvalue Fig. 4 shows the factor

\[(b(u, u') + b(u, u') - b(e, e'))^{-1}\]

for different values of \(b\). The values depend strongly on the size of \(|b|\) and eigenvalue computations beyond \(|b| > 20\) is numerically unstable. Fig. 5 compares the accuracy of the eigenvalue condition number approximation \((2b(u, u'))^{-1}\) with the error

\[\delta_i := (b(u, u') + b(u, u') - b(e, e'))^{-1} - (2b(u, u'))^{-1}\]

compared to the eigenvalue error. Since the error for the eigenvalue condition number is much smaller than the eigenvalue error for different values of \(b\), the proposed approximation \((2b(u, u'))^{-1}\) of the eigenvalue condition number is empirically efficient. In all presented numerical results the sign of \(\text{Res}_i(e')\) and \(\text{Res}_i(e)\) is in fact the same. Thus the triangle inequality \(|\text{Res}_i(e') + \text{Res}_i(e')| \leq |\text{Res}_i(e')| + |\text{Res}_i(e')|\) in the proof of Theorem 2.4 does not destroy the efficiency of the estimate.

Let \(N\) denote the number of unknowns, i.e., the number of inner nodes. Because the domain is convex, even uniform refinement results in optimal convergence rates of \(O(N^{-1})\) as shown in Fig. 6. Note that for uniform meshes \(N \approx h^{-2}\) and that there is some strong pre-asymptotic error due to the eigenvalue condition number estimate. The a posteriori error estimators \(\eta_{R, Dw2}, \eta_{R, DwM}, \eta_{R, DwA}\), and \(\eta_{DWR1}\) are close to the error while \(\eta_{R, Dw2}, \eta_{R, DwM}, \eta_{R, DwA}\) are by factors \(10^8 - 10^9\) larger than the error. Note that the first term of the error estimator \(\eta_{R, Dw2}\) is of higher order and \(\eta_{R, Dw2}\) is asymptotically reliable.

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**Fig. 3.** Primal (left) and dual (right) discrete solution for \(b = (3, 0)\) on adaptively refined meshes generated by \(\eta_{R, Dw2}\) on the unit square with about 500 nodes.

**Fig. 4.** Eigenvalue condition numbers for different values of \(b\) and sequences of uniform and adaptive meshes generated by \(\eta_{R, Dw2}\) on the unit square.
5.2. L-shaped domain

The second example is the convection–diffusion eigenvalue model problem (1.1) on the L-shaped domain

\[ \Omega = ((-1, 1) \times (-1, 1)) \setminus ([0, 1] \times [0, -1]) \]

with constant convection parameter \( \beta = (3, 0) \) and higher eigenvalues. The primal and dual solutions for adaptive meshes generated by the AFEM, based on the a posteriori error estimator \( \eta_{DWR2} \) for the 5-th eigenvalue with smallest real part, are shown in Fig. 7. An approximation of the first eigenvalue reads \( \lambda = b^2/4 + 9.6397238 \) where 9.6397238 from [30] is an approximation of the first Laplace eigenvalue. In Fig. 8 it is shown that uniform refinement results in a suboptimal convergence rate of about \( O(N^{-2/3}) \), while adaptive refinement leads to numerically optimal convergence rates of \( O(N^{-1}) \). The experiments show that the a posteriori error estimators are reliable and efficient for adaptive mesh refinement. Notice that the eigenvalues obtained from the AFEM for different estimators lead to similar eigenvalue errors. As before the values of \( \eta_{DWR2}, \eta_{DWM} \), and \( \eta_{DWA} \) are closer to the exact error than those of

Fig. 5. Eigenvalue errors and \( |\lambda| \) for different values of \( \beta \) and sequences of uniform and adaptive meshes generated by \( \eta_{DWR} \) on the unit square.

Fig. 6. Eigenvalue errors and error estimators for \( \beta = (20, 0) \) and a sequence of uniform meshes on the unit square.

5.2. L-shaped domain

The second example is the convection–diffusion eigenvalue model problem (1.1) on the L-shaped domain

\[ \Omega = ((-1, 1) \times (-1, 1)) \setminus ([0, 1] \times [0, -1]) \]

with constant convection parameter \( \beta = (3, 0) \) and higher eigenvalues. The primal and dual solutions for adaptive meshes generated by the AFEM, based on the a posteriori error estimator \( \eta_{DWR2} \) for the 5-th eigenvalue with smallest real part, are shown in Fig. 7. An approximation of the first eigenvalue reads \( \lambda = b^2/4 + 9.6397238 \) where 9.6397238 from [30] is an approximation of the first Laplace eigenvalue. In Fig. 8 it is shown that uniform refinement results in a suboptimal convergence rate of about \( O(N^{-2/3}) \), while adaptive refinement leads to numerically optimal convergence rates of \( O(N^{-1}) \). The experiments show that the a posteriori error estimators are reliable and efficient for adaptive mesh refinement. Notice that the eigenvalues obtained from the AFEM for different estimators lead to similar eigenvalue errors. As before the values of \( \eta_{DWR2}, \eta_{DWM} \), and \( \eta_{DWA} \) are closer to the exact error than those of
Fig. 7. Primal (left) and dual (right) discrete solution for $\beta = (3, 0), \lambda_0$ on adaptively refined meshes generated by $\eta_{DWR2}$ on the L-shaped domain with about 500 nodes.

Fig. 8. Eigenvalue errors and estimators for $\beta = (3, 0), \lambda_1$ and sequences of uniform and adaptive meshes on the L-shaped domain.

Fig. 9. Eigenvalue errors for $\beta = (3, 0), \lambda_1, \lambda_5, \lambda_{20}$ and $\lambda_{50}$ for sequences of uniform and adaptive meshes generated by $\eta_{DWR2}$ on the L-shaped domain.
In order to study the dependence of the a posteriori error estimators on the size of the eigenvalue, we compare the numerical results for

\[ k_1 = |\beta|^2 / 4 + 8.3713297112 \]

\[ k_5 = |\beta|^2 / 4 + 250.78548 \]

\[ k_{20} = |\beta|^2 / 4 + 101.60529 \]

\[ k_{50} = |\beta|^2 / 4 + 250.78548 \]

with approximations for the corresponding Laplace eigenvalues from [30]. Fig. 9 shows that the size of the eigenvalue error depends on the eigenvalue and that the a posteriori error estimator \( g'_{DWR1} \) is asymptotically exact. In order to investigate the dependence on the size of the eigenvalue, the efficiency indices \( I_{\text{Eff}} = g' / \lambda / C_0 \) for \( k_1 \) and \( k_{50} \) are compared in Fig. 10. The experiments show that the ratio between the a posteriori error estimators and the eigenvalue error is growing in \( k \) for \( g'_{R}, g'_{A}, \) and \( g'_{DWR1} \) while \( g'_{DWR2}, g'_{DWM}, \) and \( g'_{DWA} \) are robust in \( \lambda \). Note that the efficiency indices of \( g'_{DWR2}, g'_{DWM}, \) and \( g'_{DWA} \) are close to one.

5.3. Slit domain

As last example consider the convection-diffusion eigenvalue model problem (1.1) on the slit domain \( \Omega = ((-1,1) \times (-1,1)) \setminus ([0,1] \times (0)) \) with different constant values for \( \beta \). A computed reference value for the first eigenvalue reads \( \lambda = |\beta|^2 / 4 + 8.3713297112 \) with approximation 8.3713297112 of the first Laplace eigenvalue computed on very fine meshes and higher order finite elements. The primal and dual eigenfunctions on adaptive meshes for \( g'_{DWA} \) are shown in Fig. 11. Notice that for the primal eigenfunction the influence of the magnitude of the corner singularity at the origin is much larger than for the dual eigenfunction. This illustrates that it is important to consider both primal and dual residuals. Due to
Fig. 11. Primal (left) and dual (right) discrete solution for $\beta = (3, 0)$ on adaptively refined meshes generated by $\eta_{DWA}$ on the slit domain with about 500 nodes.

Fig. 12. Eigenvalue errors and estimators for $\beta = (15, 0)$ and sequences of uniform and adaptive meshes on the slit domain.

Fig. 13. Meshes with $\beta = (15, 0)$ generated by the refinement monitored by $\eta_{R}, \eta_{A}, \eta_{DWR1}, \eta_{DWR2}, \eta_{DWM}$ and $\eta_{DWA}$ (from left to right and top to bottom) on the slit domain with about 2500 nodes.
the corner singularity, uniform refinement results in a suboptimal convergence rate $O(N^{-1/2})$ while adaptive refinement results in the optimal convergence rate $O(N^{-1})$ as shown in Fig. 12 for $\beta = (15, 0)$. Note that the eigenvalue errors for $\eta_{DWR}$ and $\eta_A$ are much larger than for $\eta_{DWR1}, \eta_{DWR2}, \eta_{DWM}$ and $\eta_{DWA}$ and even larger than the eigenvalue errors for uniform refinement up to $N_r = 10^6$. This observation is caused by a much larger pre-asymptotic range for $\eta_{DWR}$ and $\eta_A$ than for the DWR based a posteriori error estimators. The different adaptive meshes with about $N_r \approx 2500$ are shown in Fig. 13. The meshes for $\eta_{DWR}$ and $\eta_A$ show strong refinement towards the two boundary layers on the left and right but almost no refinement towards the corner singularity at the origin which might cause the larger eigenvalue errors. In contrast to that all other refinement indicators show strong refinement toward the corner singularity at the origin which leads to smaller eigenvalue errors. In order to study the dependence of the a posteriori error estimators on the size of the convection parameter, experiments for $\beta = (1, 0)$ and $\beta = (15, 0)$ are compared in Fig. 14. The constants of the estimates in Lemma 3.1 and Lemma 3.2 depend on the size of the convection parameter. Thus, the efficiency indices $I_{Eff}$ are expected to depend on the size of $|\beta|$ as well which is confirmed by the numerical experiments. The size of the efficiency indices grows for the a posteriori error estimators $\eta_{DWR1}, \eta_A$ and $\eta_{DWR2}$ corresponding to the increase of $|\beta|$. In contrast the efficiency indices for $\eta_{DWR2}, \eta_{DWM}$ and $\eta_{DWA}$ are robust in $\beta$ and asymptotically close to one.

6. Conclusions

All the numerical results indicate that the a posteriori error estimators are empirically reliable and efficient for sufficiently small global mesh-size. The interpolation scheme of Section 4 for the weights shows to be empirical stable for
unstructured triangular meshes. The approximation of the condition number needs to be included in the a posteriori error estimators in order to get efficiency indices close to one. The DWR2, DVM and the DWA a posteriori error estimators result in the best asymptotic efficiency indices close to one independently of both, the size of the eigenvalue and the convection parameter. For larger values of $|\beta|$ the DWR based a posteriori error estimators show much better results than the residual and averaging based a posteriori error estimators because of the much smaller pre-asymptotic range. Since the used eigenvalue solver ARPACK [21] shows some instability for convection coefficients larger than (20,0) and coarser meshes those are excluded in this paper. For highly non-symmetric problems other techniques such as homotopy methods [10] need to be applied in order to compute the same eigenvalue of interest during all steps of the adaptive finite element loop or different finite elements need to be considered such as discontinuous Galerkin finite elements [11].

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